Modeling Sustainable Agricultural Production Systems

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Abstract

This paper proposes a model a sustainable agricultural production System. We show how to characterize the overall production process using a suitable distance function. Along this line, we elaborate a linear programming model to measure the maximum expansion of the crop-output matrix and the terminal soil-capital vector.

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1 Introduction

Our approach to modeling sustainable agriculture is rooted in the natural resource economics literature. We view agricultural production as a fundamentally dynamic process in which one year's production is inextricably linked with the future capacity of the land to produce other crops and products; in other words, agriculture is essentially a resource-based industry. Climate–the seasonal and stochastic distribution of rainfall and temperatures–is an obviously important resource, providing such well-recognized inputs as moisture and energy as well as less-recognized services such as insect, weed, and disease control from freezing; but these resources are generally not susceptible to human intervention and are thus excluded from our analysis. Within a given climatic zone (production region), the resource base of agriculture consists of a set of attributes largely embodied in the quantity and quality of soil. Soil in particular provides a wide variety of services that are critical in the production of agricultural crops, including structural support for plants; storage and transport of water, oxygen, and nutrients; and habitat for beneficial organisms, such as nitrogen-fixing bacteria and earthworms. Taken together, these physical, chemical, and biological attributes of soil constitute what we will term soil capital. We consider soil capital as the crucial natural resource in agriculture, at least in the Mid-Atlantic and similar regions where rain-fed farming predominates. Models of agricultural production systems need to take into account their inherently dynamic nature, or, put another way, net investment in soil capital. Depletion of soil capital in an effort to increase current profitability or productivity impinges upon the farmer's future ability to pursue profitability or productivity at later dates. Similarly, rotational systems that preserve or enhance soil capital may effectively forego output or profitability in the current period to enhance output or profitability in the future. Simply put, the farmer must recognize that besides being able to substitute between inputs and outputs in the current period, one also has the ability to substitute intertemporally as well. The conduit for this intertemporal substitution is soil capital. Enhancing soil capital essentially constitutes an investment in future productivity. Recognition of the critical importance of soil capital in agriculture has several implications for modeling agricultural technologies. First, in more formal production theoretic terms, soil capital is an intermediate input in an intertemporal production technology. In other words, in addition to crops, livestock, and other outputs produced on the land, agricultural production processes create a series of inputs used in the production of other crops and outputs on the same parcel of land in future periods. Because, as we have seen, these inputs are largely embodied in soil, we refer to them generically as soil capital. Second, farming systems models must be explicitly dynamic and multi-output, based on crop rotations rather than individual crops. As an intermediate input, soil capital is produced jointly with crops, livestock, and other outputs in one period and affects outputs of crops and livestock in future periods. Recognizing this fact implies a need for dynamic multi-output models. Modelling crop rotations is essential because the central thrust of self-identified sustainable or alternative agriculture is replacing purchased inputs like chemical fertilizers or pesticides with investment in soil capital through crop rotation, green-manure crops, increased labor, and intensified management.

Harwood characterizes this approach as one of farming the organic matter fraction of the soil rather than farming the soil-nutrient solution. Third, investment in soil capital encompasses both physical and biological processes. The biological processes are arguably the most important, at least on a human time scale, as Harwoods characterization attests. Experience with pesticides (Lichtenberg and Zilberman; Babcock, Lichtenberg and Zilberman; Chambers and Lichtenberg, 1994) and fertilizers (Ackello-Ogutu, Paris and Williams; Grimm, Paris and Williams; Frank, Beatty, and Embleton; Paris; Chambers and Lichtenberg, 1995) indicates that these biological factors give rise to production relationships that are highly nonlinear. Recognizing that pesticides protect crops rather than enhance crop growth suggests that pesticide productivity is bounded by a maximum possible output given application of normal inputs and weather conditions. The von Liebig hypothesis about plant growth also suggests that there may be limited substitutability among nutrients and that nutrients and other inputs may be limitational1: When limitational inputs are held constant, the marginal productivity of other inputs can be, i.e., there may exist yield plateaus. We also follow the natural resource economics literature in defining a sustainable production system as one that produces a non-declining net value of output over time. We choose net value of output as an indicator of well-being because it measures actual value in consumption. In the more general context of long-run growth, Weitzman (1976) has shown that the present value of net output in any given year, which consists of current consumption expenditures plus net investment (investment less depreciation), equals the present value of the optimal stream of consumption plus net investment, so that current-year net output is a good measure of well-being in an intertemporal context. In cases where natural resources are important–as soil capital is in agriculture–the concept of net investment must be expanded beyond traditional investment in manufactured capital to include investment in and/or depletion of natural resources, as work by Solow, Hartwick (1990, 1991), Dasgupta et al., and others has demonstrated. Thus, a central aim of our modeling effort will be to define appropriate measures of net investment in soil capital and of the value of the soil-capital stock. Adjustments for environmental quality are also typically needed in cases where pollutant emissions exceed the environments natural absorptive capacity. In some cases, the environment enters social well-being either as a flow of consumption goods or services; in other cases, it is more appropriately dealt with as a form of capital. Weitzman (1994) and Hartwick (1990) discuss adjustment of net-output measures to incorporate environmental quality in these two respective cases. However, incorporating environmental quality effects into empirical models is typically difficult. Data on farming systems come from agronomic experiments that are typically not designed with measurement of environmental effects in mind. Environmental impacts tend to vary markedly from farm to farm even within a narrow region, because they depend on location-specific factors such as proximity to surface water or ground water recharge areas, extent of wildlife habitat in areas surrounding the farm, and so on. They are also critically dependent on stochastic factors; for example, nutrient leaching in the Mid-Atlantic generally occurs during large rainfall events on land with no vegetative cover. For these reasons, environmental effects will be largely outside the scope of our modeling effort. Many sustainable farming systems may co-exist in a given region, and conventional systems may be sustainable. In other words, many systems may be capable of generating a non-declining flow of net output over time; and even though use of chemicals synthesized from exhaustible resources like petroleum may not be sustainable forever, the date at which exhaustibility begins to matter may be so far off that it can be safely ignored. Moreover, not all conventional farming systems are based on depletion of soil capital. Many modern soil and water-conservation techniques use manufactured inorganic fertilizers and pesticides to invest in soil capital. Conservation tillage, which maintains soil depth and structure by substituting herbicides for deeper and more frequent tillage, is one example. Integrated pest management, which combines cultural and biological controls with judicious use of chemical pesticides, is another. The sustainability of conventional agricultural production systems is thus of considerable interest. In what follows, we shall attempt to develop a formal theoretical model of such a rotational production process that faithfully replicates what we know about intertemporal agricultural production processes. We start by laying out the notation that we shall use.

2 Notations

Let $\tau \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, denote the number of years in a rotation. $y \in \mathbb{R}^{\tau \times m}$ represents the matrix of outputs produced in the rotation. To be specific, y_{it} denotes amount of the i^{th} output produced in the t^{th} year of the rotation. $x_{it} \in \mathbb{R}^{\tau \times m}$ is a matrix of time dated variable inputs whose typical element x_{kt} denotes the amount of input k applied in production year t. A comment is appropriate here: For some of the rotations observed in the Rodale FST, different inputs are applied depending upon the point one is in the rotation. For example, seed for a particular cover crop may not be sown in year one of a particular rotation (i.e., the rotation begins with crop production, not field preparation), but may be sown in all succeeding years. The row dimension of x (y), however, does not change as we move through the production rotation. If an input is used in one year of the production cycle but not in others, it simply enters x as a zero in the years it is not used. Let $s \in \mathbb{R}^{\tau \times q}$ be a matrix of time-dated soil-capital attributes that make up capital.[1] Let x_t denote the t^{th} row of x, and denote y_t and s_t. in a similar fashion. s_0 denotes the vector of soil capital observations at the start of the sample period.

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[1] Our convention is to measure soil capital at the end of the production year and not at the beginning of the production year. This is in accordance with our interpretation of soil capital as an intermediate input.

3 A General Representation of a Sustainable Agricultural Technology

Denote by $vec(x)$ the row vectorization of x, i.e., $vec(x) = (x_1, x_2, ..., x_\tau)$, and let us denote $X = vec(x)$. Use a similar definition for $S = vec(s)$ and $Y = vec(y)$, and let $S_{-0} = vec(s_{-0}) = (s_0, s_1, ..., s_{\tau-1})$. The general technology views input k applied at time t as a distinct input from input k applied at time t^* with $t \neq t*$. Moreover, as we discussed above, any representation of the technology must also recognize that soil capital is not a traditional variable input in the sense that at each point in time the producer can freely choose the levels of soil capital. Instead, soil capital is a both an input and an output in the sense that soil capital at time $t-1$ interacts with inputs applied at time t to affect (presumably positively) crop production (y_t) in period t, while, at the same time, soil capital at the end of the production period, s_t, is determined by cropping patterns and the rates at which other inputs are applied during time t. Hence, it is appropriate to treat (y_r, s_t) as joint products of inputs (x_t, s_{t-1}), recognizing, in effect, that both the crop output and soil capital are produced during the t^{th} production period. These production relations are summarized by an overall production process defined by:

$$\mathcal{T} = \Big\{(X, S, Y) \in \mathbb{R}_+^{\tau\times(n+m+q)} : (X, S_{-0}) \text{ can produce } (Y, S)\Big\}. \tag{3.1}$$

In words, $\mathcal{T}$ just represents all combinations of variable inputs, intermediate soil-capital outputs, and crop outputs which are technically feasible, that is, $\mathcal{T}$ is a standard Arrow-Debreu technology consisting of time-dated inputs and outputs[2]. Our primary task in this paper is to develop a set of suitable regularity properties for T which allows us to develop an operational model of a rotational production process.

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[2] Notice that $\tau - 1$ elements of S and S_{-0}) overlap and thus do not vary independently. T could be represented in a more compact notation that does not emphasize the role that soil-quality plays as an intermediate output by eliminating this overlap. For example, we could express T alternatively as being the set defined by:

$$\mathcal{T} = \big\{(X, Y, S, S_0) : (X, S_{-0}) \text{ can produce } (Y, S)\big\}.$$

We choose the current representation to emphasize the notion that soil capital is produced, and because it is also notationally convenient for the temporally separable technology developed below.

Perhaps the first thing to notice about this specification of the technology is that even though we have not imposed any specific regularity properties (for example, convexity, free disposability of inputs, etc.) upon $\mathcal{T}$, $\mathcal{T}$ is implausible in the sense that it allows variable inputs applied in year $t+k$, say, to have a direct impact upon outputs produced in year t, i.e., inputs can affect production before they are applied. Similarly, output in year $t+k$ can have a direct effect on what happens in year t. Or put more simply, T allows the future to determine the past. The possibility of backwards temporal causality is not peculiar to our formulation; rather, it is a standard feature of an Arrow-Debreu technology. Although this may be a plausible property to impose upon some economic models, it most certainly is not a plausible property to impose upon biological production processes as we currently understand them to operate. An alternative specification that does not allow temporal relationships generally considered impossible is what we shall term the temporally separable model of agricultural production. The temporally separable model labors under the obvious premise that the future cannot cause the past, i.e., what happens in period $t+k$ cannot have a direct effect upon period t while still allowing the past to affect the future. Basic biological information suggests that production in a sustainable rotation operates in the following manner: At time period t, variables inputs applied during the production period, x_t , interact with nutrients contained in the soil (as measured by s_{t-1}) to produce the vector of agricultural outputs at the end of the production period, y_t , and soil quality at the end of period t as measured by st. That is, at time period t, we have a distinct production set:

$$T_t = \left\{(x_t, y_t, s_{t-1}, s_t) \in \mathbb{R}_+^{(n+m+2q)} : (x_t, s_{t-1}) \text{ can produce } (y_t, s_t)\right\}. \tag{3.2}$$

The most important characteristic of T_t is that each period's production set only depends upon what has happened previously to the extent that previous input and output decisions have affected soil capital at the beginning of the production period. The overall production process for the entire rotational cycle is thus represented by:

$$\mathcal{T} = \left\{(X, S, Y) \in \mathbb{R}_+^{\tau\times(n+m+q)} : (x_t, y_t, s_{t-1}, s_t) \in T_t, t = 1, ..., \tau\right\}. \tag{3.3}$$

$\mathcal{T}_{\mathcal{O}}$ is a biologically correct representation of the underlying technology so long as the set of soil capital variables S is exhaustive, that is, as long as s_{t-1} contains all the information about

$s_{t-2}, ..., s_0$ that has any bearing on production in period t or any later period. Empirically, this means that the components of soil capital we observe must contain all the relevant information about the effects of crop production and input use in the years prior to $t-1$ on output of crops and soil capital in year t and all later years. In theory, then, the temporally separable representation is identical to the general Arrow-Debreu representation in all respects save the restriction ruling out backward temporal causation.

4 Properties of Agricultural Technologies

Even though $\mathcal{T}$ represents a substantial specialization of $\mathcal{T}$, it is not a workable representation of a sustainable agricultural technology without further restrictions (axioms) being placed upon it. In considering properties that one may wish to impose upon a sustainable technology, a natural starting place is the list of properties that are imposed upon most empirical representations of agricultural technologies. These would include eventually diminishing marginal productivity and nonnegative marginal productivity. In the following, we briefly consider a series of axioms which serve as a menu of properties that one might impose upon the sustainable agricultural technology.

We emphasize that they represent a menu of properties. In almost all instances, our empirical study only imposes a subset of them at any one time. Several alternative representations of the technology will facilitate the discussion. The output set, which consists of all outputs that are technically feasible for a given set of inputs, for T_t is defined by the correspondence:

$$P_t(x_t, s_{t-1}) = \Big\{(y_t, s_t) : (y_t, x_t, s_{t-1}, s_t) \in T_t\Big\}. \tag{4.1}$$

In intuitive terms, the output set represents all output combinations that are on or below a production-possibilities frontier. Dual to the output set is the input set, consisting of all input combinations capable of producing a given vector of outputs which for T_t is defined by the correspondence:

$$L_t(y_t, s_t) = \Big\{(x_t, s_{t-1}) : (y_t, s_t) \in Y_t(x_t, s_{t-1})\Big\}. \tag{4.2}$$

Intuitively, the input set generalizes the traditional notion of an isoquant to include all levels of inputs, including possibly inefficient ones, which can produce a given vector of outputs. From the way that we have defined the input and output sets, it is apparent that:

$$(y_t, s_t) \in P_t(x_t, s_{t-1}) \iff (x_t, s_{t-1}) \in L_t(y_t, s_t) \iff (y_t, x_t, s_{t-1}, s_t) \in T_t. \tag{4.3}$$

In words, this last mathematical statement simply says that any of the three representations of the period t agricultural technology is equivalent to any of the remaining two representations. At a practical level, this allows us to work with whichever representation of the technology turns out to be the most convenient.[3] (In most instances, for the Rodale FST data, this will turn out to be the output set.) The input and output sets for T and $\mathcal{T}$ are defined similarly. Here we list a set of axioms that we wish to consider imposing upon the output set. Once they have been listed, we describe them intuitively.

$P.1$: $(0_m, 0_q) \in P_t(\mathbb{R}^{n+q})$, $(y_t, s_t) \notin P_t(0_n, 0_q)$ for $(y_t, s_t) \neq (0_m, 0_q)$, $t = 1, 2, ..., \tau$;
$P.2$: For all (x_t, s_{t-1}), $P_t(x_t, s_{t-1})$ is nonempty and closed;
$P.3$: $\theta P_t(x_t, s_{t-1}) + (1-\theta)P_t(x'_t, s'_{t-1}) \subset P_t(\theta(x_t, s_{t-1}) + (1-\theta)(x_t, s_{t-1}))$, $0 \leq \theta \leq 1$;
$P.4$: $(x'_t, s'_{t-1}) \geq (x_t, s_{t-1}) \Longrightarrow P_t(x_t,' s_{t-1}) \subset P_t(x'_t, s'_{t-1})$;
$P.5$: $(0_m, 0_q) \leq (y'_t, s'_t) \leq (y_t, s_t) \Longrightarrow P_t(x_t, s_{t-1}) \Longrightarrow (y'_t, s'_t) \in P_t(x_t, s_{t-1})$.

Axiom P.1 has two parts. The first says that one can always choose to produce nothing regardless of the amounts of variable inputs and soil capital available. The second, which is sometimes referred to as a no-free-lunch assumption implies that any positive vector of crop outputs cannot be produced without some committal of variable inputs or soil capital. As such, the latter part of P.1 is a particularly weak assumption (and presumably, therefore, believable) which could be strengthened in a number of plausible ways. For example, one could reasonably require that soil capital be a strictly essential input, i.e., if soil capital is absent (here we presume that units of measurement for soil capital are chosen so that a zero observation indexes some biologically determined level which is insufficient to support plant growth) the production of crop output and soil capital is impossible. Generally, even

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[3] In more formal discussions of technical relationships T_t is often referred to as the graph of the technology, see e.g. Färe.

though strict essentiality of soil capital is a believable assumption, it may be overly strong in some empirical contexts. For example, if observations on s_t are incomplete in the sense of not including all of the components of soil capital contributing to crop growth, then the observed subset of components of soil capital will not generally be strictly essential. Therefore, in what follows we shall rely instead upon the weaker version contained in P.1. Axiom P.2 says that some bundle of crop outputs and soil capital is producible. It implies that some T_t is nonempty. It does not imply, however, that every conceivable output set is nonempty. As such, it is an assumption which cannot be contradicted by any real-world data set which has positive observations on soil capital and crop output, i.e., any real-world data set which we are likely to be interested in. The second part of Axiom P.2 is a purely mathematical assumption which assures the existence of maxima in some of the analysis that follows. It, too, cannot be contradicted by any body of observable data. Axiom P.3 says that all of these output combinations are producible using some weighted average of (x_t, s_{t-1}) and (x'_t, s'_{t-1}). Axiom P.3, therefore, makes several important assumptions. First, it presumes that one can, in fact, take a weighted average of arbitrary combinations of inputs. Inputs, therefore, must be perfectly divisible. Some inputs like tractors are obviously not physically divisible, but the services they provide (e.g., tractor-hours) may well be conceived of as perfectly divisible. However, it is also possible that scheduling problems or other factors limit the extent to which even these services are perfectly divisible, at which point this restriction becomes unrealistic. Second, Axiom P.3 implies that the input set and the output set are convex sets. Convexity of the output set implies that if soil capital and the crop output or any two outputs are substitutes for one another, they substitute for one another only at increasing marginal cost. Put another way, if the only path to increasing soil capital from a fixed bundle of inputs is to cut back on production of crop output, Axiom P.3 implies that each marginal unit of increase in soil capital is purchased only at the expense of increasingly large sacrifices of crop. Note that Axiom P.3 does not imply that crop output and soil capital cannot be complements, that is, it does not rule out simultaneous accretion of soil capital and rising yields.

Third, Axiom P.3 has a similar implication on the input side: Inputs exhibit diminishing marginal rates of technical substitution. And finally, Axiom P.3 implies that inputs exhibit diminishing marginal productivities. In fact, in the case of a scalar output, P.3 corresponds to the presumption that the production function, if one exists, is concave in inputs. In what follows, we shall always maintain P.2 - P.3. As we have tried to make clear, P.2 seems relatively uncontroversial to us, and while P.3 has some obvious shortcomings, e.g. perfect divisibility, it is a standard assumption that in applied production analysis. However, if Axiom P.1 is always maintained in conjunction with Axiom P.3, it implies that the production technology is subhomogeneous, i.e., there can never exist any increasing returns to scale. Therefore, Axiom P.1 will not always be deployed. Turning to Axioms P.4 and P.5: Both of these assumptions are routinely maintained in applied production analysis, but we feel that either or both may not particularly appropriate for a sustainable agricultural technology. We shall address them in reverse order, considering P.5 first. Suppose that input bundle $(x_{t\cdot}, s_{t-1})$ is capable of producing the soil-capital, crop-output combination A. P.5 then implies that this same input bundle can produce any output bundle that lies to the southwest of A. In other words, outputs can be disposed of without imposing any cost in terms of foregone output, hence the name free disposability. For many technologies, this may not seem to be an overly restrictive assumption because, intuitively it only requires you to use the same input bundle in an inefficient manner. Notice however, But when it is coupled with P.3, it has the implication that any two outputs, when considered alone, can never have a complementary relationship. Now consider the implications of this assumption for a sustainable agricultural technology. It implies that there cannot be any points on the frontier of the output set which are consistent with soil capital and output varying together, i.e., exhibiting complementarity. In other words, Axiom Y.5 means that increases in output necessarily requires disinvestment in soil capital. But one important contention in the literature on alternative agriculture is that output and investment in soil capital may be complementary. For example, one might expect greater nitrogen fixation when yields of legume crops are higher. Similarly, one would expect greater augmentation of soil organic matter from crop residue when crop yields are higher. It thus seems likely that, at least over certain ranges, there are ways of rearranging inputs that will result in simultaneous increases in soil capital and crop output. Now consider Axiom P.4. Economically, it implies that any marginal increment of a variable input when applied to an otherwise fixed input bundle never reduces the range of outputs that the fixed input bundle can produce. Put another way, it implies that all marginal productivities are nonnegative. Practical experience suggests that there are situations where marginal application of an input to an otherwise fixed bundle of inputs results in a fall in marginal productions. The most familiar explanation for why this occurs is a 'congestion' effect: Too much of the variable input is applied for the remaining fixed inputs to be able to cooperate with efficiently. A stark, but still illustrative example, should make this a bit clearer. Suppose the variable input that we are talking about is water. Clearly moisture is necessary for plant growth, but just as clearly too much water applied to a given plot of land can be deadly. The operative phrase in the last sentence is "...to a given plot of land". Or, put another way, the problem isn't too much water so much as it is too little land. If the amount of land (as well as other fixed inputs) had been varied in proportion to the amount of water applied, the congestive effect of the marginal application of water would not have occurred.

Obviously, the specifications above have some strong implication on the overall production process. On can use a symmetrical characterization of the output sets regarding to the technology $\mathcal{T}$. This we do by defining the output correspondence

$$\mathcal{P}(X, S_{-0}) = \Big\{(Y, S) : (X, S, Y) \in \mathcal{T}\Big\}. \tag{4.4}$$

5 Functional Representations of the Sustainable Technology

When pressed to describe a productive relationship in technical terms, most economists revert to some functional representation of the technology, most typically the production function. There are many advantages, both intuitive and purely technical, to thinking in terms of the production function: Most importantly, it is a functional representation that exhaustively characterizes the technology. However, the production function is expressly defined for a technology with a single output. If we are to have a functional representation of the technology that adequately captures the multi-faceted output relationships that characterize sustainable agricultural technologies, we need to think in some other terms than the production function. One natural extension is the asymmetric transformation function which singles out one particular output, holding all other outputs and input fixed, and then maximizes the amount of the output produced. This approach does yield a functional relationship that exhaustively characterizes the production technology. However, it is stylistically inelegant because it treats outputs asymmetrically, i.e., treating one as an output and the others essentially as inputs. Moreover, this approach is generally implemented using statistically-based parametric representations, which are not always feasible given the data commonly available. An alternative functional representation of the technology is an output distance function. We develop characterizations of agricultural technologies using the output distance function for two main reasons. First, the output distance function is perhaps the most natural generalization of the production function familiar from standard economic theory (our brief discussion of it below shall try to emphasize this fact); and, second, it is better suited to modeling this specific problem. There are at least three output directional distance functions that we will be interested in examining empirically.

We first consider a reference crop matrix $G_C = vec(g_{C,1}, ..., g_{C,\tau}) \in \mathbb{R}_+^{\tau \times m}$. We suppose that $G_C \neq 0_{\tau \times m}$. Following, Chambers, Chung and Färe (1996,1998), we first define *the crop-output directional distance function*:

$$\mathcal{D}_C(X, Y, S; G_C) = \sup \Big\{ \delta : (Y + \delta G_C, S) \in \mathcal{P}(X, S_{-0}) \Big\}. \tag{5.1}$$

Equivalently, on has:

$$\mathcal{D}_C(X, Y, S; G_C) = \sup \Big\{ \delta : (X, Y + \delta G_C, S) \in \mathcal{T} \Big\}. \tag{5.2}$$

Visually, $\mathcal{D}_C$ corresponds to the largest translation in the direction of G of the crop output matrix that is technically feasible given the trajectory of soil capital and variable inputs.

We have seen that axiom $P.5$ may not particularly appropriate for a sustainable agricultural technology. Hence, we slightly weaken them using the direction of G.

$P'.5$ For all $\delta \leq 0$, $(y_t, s_t) \in P_t(x_t, s_{t-1})$ and $y_t - \delta g_{C,t} \geq 0_m \Longrightarrow (y_t - \delta g_{C,t}, s_t) \in P_t(x_t, s_{t-1})$.

Axiom $P'.5$, which is usually referred to as weak output disposability, permits a limited form of complementarity between soil quality and crop output. Where $P.5$ guarantees that any point to the southwest of point on the boundary of the output set is producible from the same input bundle as that used to produce the boundary point, $P'.5$ implies that only contractions in the direction of g_t of boundary points need be producible using the input bundle as the boundary point.

$\mathcal{D}_C$ has a number of interesting properties that are of particular importance to specialists in production economics, but the most important for our purpose here is that provided that $\mathcal{T}$ satisfies $P'.5$ (which we enumerate below), then Dc is a complete functional representation of the technology in that:

$$(X, S, Y) \in \mathcal{T} \iff \mathcal{D}_C(X, S, Y; G_C) \geq 0. \tag{5.3}$$

To see why the equivalence follows, first suppose that $(X, S_{-0}, Y, S) \in \mathcal{T}$ but that $\mathcal{D}_C(X; Y, S; G_C) > 0$. Clearly, this last inequality violates the definition of the crop-output distance function. To see the converse, suppose that $\mathcal{D}_C(X, Y, S; G_C) > 0$, the above notion of weak disposability then implies that $(X, S_{-0}, Y) \in \mathcal{T}$.

The second output distance function that we shall consider is what we shall simply refer to as the output directional distance function and it represents the largest expansion of crop output and soil capital in a direction $G = (G_C, G_S) \in \mathbb{R}_+^{\tau\times m} \times \mathbb{R}_+^{\tau\times m}$, consistent with a specified trajectory of variable inputs and initial soil capital, i.e.,

$$\mathcal{D}(X, Y, S; G) = \sup\Big\{\delta : (Y + \delta G_C, S + \delta G_G) \in \mathcal{P}(X, S_{-0})\Big\}. \tag{5.4}$$

Equivalently, on has:

$$\mathcal{D}(X, Y, S; G) = \sup\Big\{\delta : (X, Y + \delta G_X, S + \delta G_S) \in \mathcal{T}\Big\}. \tag{5.5}$$

The function above can be useful to characterize the technology. However, this requires a slight change of axiom $P'.5$.

$P''.5$ For all $\delta \leq 0$, $(y_t, s_t) \in P_t(x_t, s_{t-1})$, $(y_t - \delta g_{C,t} \geq 0_m$ and $s_{t-1} - \delta g_{X,t}) \geq 0 \implies (y_t - \delta g_{C,t}, s_t - \delta g_{S,t}) \in P_t(x_t, s_{t-1})$.

Given the natural extension of weak disposability axiom P".5 to include weak disposability of soil-capital outputs, $\mathcal{D}_T$ offers a complete function representation of the technology in the sense that[4]:

$$(X,Y,S) \in \mathcal{T} \iff \mathcal{D}(X,Y,S;G) \leq 0. \tag{5.6}$$

It is obvious to see that

$$\mathcal{D}(X,Y,S;G) \leq \mathcal{D}_C(X,Y,S;G_C). \tag{5.7}$$

The final output distance function that we shall be interested in is what we shall refer to as the terminal-value output distance function which describes the maximum radial expansion of the crop output matrix and terminal soil-capital vector consistent with the technology for a given trajectory of soil-capital up to the beginning of period τ:

$$\mathcal{D}_{T,\tau}(X,Y,S;G) = \sup\Big\{\delta : (Y+\delta G_C, s_1, ..., s_{\tau-1}, s_\tau + \delta g_{S,\tau}) \in \mathcal{P}(X,S_{-0})\Big\} \tag{5.8}$$

$\mathcal{D}_{T,\tau}$ also offers a complete function representation of $\mathcal{T}$ under an appropriate version of weak disposability of outputs. However, what is more important is that $\mathcal{D}_{T,\tau}$ also provides a natural means for computing a virtual price of soil capital in the last period of the rotation. When the technology is temporally separable, it becomes particularly easy to compute the crop-output distance function and terminal-value output distance function. Denote:

$$D_C^t(x_t, y_t; s_t, s_{t-1}; g_{C,t}) = \sup\Big\{\delta \geq 0 : (y_t + \delta g_{C,t}, s_t) \in P_t(x_t, s_{t-1})\Big\} \tag{5.9}$$

$$D^t(y_t, s_t; x_t, s_{t-1}; g_t) = \sup\Big\{\delta \geq 0 : (y_t + \delta g_{C,t}, s_t + \delta g_{C,t}) \in P_t(x_t, s_{t-1})\Big\}, \tag{5.10}$$

where $g_t = (g_{C,t}, g_{S,t})$. With this notation we can now show that[5]:

Proposition 5.1 *If the technology is temporally separable then:*

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[4] This version of the output distance function implies that s- 0 and s can be varied independently of one another which cannot happen in a closed rotational system. However, because in our empirical analysis we only calculate distance functions at observed data points, this limitation is not of crucial importance to this study.

[5] We wish to acknowledge helpful conversations with Ted Jaenicke on this point.

$$\mathcal{D}_C(X;Y,S;G_C) = \inf_{t=1,\ldots,S} D^t_C(x_t;,y_t,s_t,s_{t-1};g_{c,t})$$

and

$$\mathcal{D}_{T,\tau}(X,Y,S;G) = \min\Big\{\min_{t=1,\ldots,\tau-1} D^t_C((x_t,y_t,s_t,s_{t-1};g_{C,t}), D^\tau(x_\tau,y_\tau,s_\tau,s_{\tau-1};g_\tau)\Big\}.$$

Proof: We only prove the second expression. The first expression can be proved in an entirely analogous fashion. If the technology is temporally separable, then by definition:

$$\begin{aligned}\mathcal{D}_T(X,Y,S;G) = \sup\Big\{\delta \geq 0 :&(y_t+\delta g_{C,t},s_t) \in P_t(x_t,s_{t-1}), t=1,2,...,\tau-1,\\ &(y_\tau,s_\tau)-\delta g \in P_\tau(x_\tau,s_{\tau-1})\Big\}.\end{aligned}$$

Therefore, it must be true that

$$\big(y_t - \mathcal{D}_{T,\tau}(X,Y,S;G)g_{C,t},s_t\big) \in P_t(x_t,s_{t-1}),$$

for $t = 1,2,...,\tau-1$ and

$$(y_\tau - D_{T,\tau}(X,Y,S;G)g_{C,\tau}, s_\tau - D_{T,\tau}(X,Y,S;G)g_{S,\tau}) \in P_\tau(x_\tau,s_{\tau-1}),$$

whence

$$\mathcal{D}_T(X,Y,S;G) \geq D^t_C\big(x_t;y_ts_t,s_{t-1};g_{C,t}\big),$$

for $t = 1,2,...,\tau-1$ and $\mathcal{D}_T(X,Y,S;G) \geq D^\tau(x_\tau,s_\tau;s_S\tau,s_{\tau-1};g_\tau)$. The conclusion follows immediately.□

This Proposition considerably eases the computational burden associated with calculating output distance functions for temporally separable technologies.[6]

6 A Linear Programming Model

The non parametric approach (see for instance Varian (1984) and Banker, Charnes and Cooper (1984)) can be useful to estimate a production technology. In such a case the

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[6] We wish to acknowledge helpful conversations with Ted Jaenicke on this point.

technology is, loosely speaking, the weakly monotonic convex hull of a finite set of observed production vectors. Notice that the following approach impose the strong disposability assumption. Let $\mathcal{J} = \{1, \cdots, J\}$ be a set of firms and let,

$$A_t = \Big\{(x_{j,t}, s_{j,t-1}, y_{j,t}, s_{j,t}) : j \in \mathcal{J}\Big\} \tag{6.1}$$

be the corresponding set of observed production vectors. This means that for all $j \in \mathcal{J}$, we are able to observe the matrices $X_j = vec(x_{j,1}, ..., x_{j,\tau})$, $Y_j = vec(y_{j,1}, ..., y_{j,\tau})$ and $S_j = vec(s_{j,1}, ..., s_{j,\tau})$. Moreover, we denote $S_{j,-0} = vec(s_{j,0}, ..., s_{j,\tau-1})$. Adopting an convexity assumption (P.3) we can suppose that:

$$\begin{aligned}\hat{T}_t = \Big\{(x_t, s_{t-1}, y_t, s_t) \in \mathbb{R}_+^{n+m+2q} \colon (x_t, s_{t-1}) \geq \sum_{j \in \mathcal{J}} \theta_j (x_{j,t}, s_{j,t-1}),\\ (y_t, s_t) \leq \sum_{j \in \mathcal{J}} \theta_j (y_{j,t}, s_{j,t}), \sum_{j \in \mathcal{J}} \theta_j = 1, \theta \geq 0\Big\}.\end{aligned} \tag{6.2}$$

Hence, the directional distance function is calculated solving the program:

$$\begin{aligned}D_C^t(x_t, s_{t-1}, y_t, s_t; g_{C,t}) = \max \Big\{\delta : (x_t, s_{t-1}) \geq \sum_{j \in \mathcal{J}} \theta_j (x_{j,t}, s_{j,t-1}),\\ (y_t + \delta g_{C,t}, s_t) \leq \sum_{j \in \mathcal{J}} \theta_j (y_{j,t}, s_{j,t}), \sum_{j \in \mathcal{J}} \theta_j = 1, \theta \geq 0\Big\}.\end{aligned} \tag{6.3}$$

If the directional distance functions is crop-oriented, this yields the following linear program:

$$\begin{aligned}D_C^t(x_t, s_{t-1}, y_t, s_t; g_{C,t}) = {} & \max \delta\\ & (x_{i,t}, s_{i,t-1}) \geq \sum_{j \in \mathcal{J}} \theta_j (x_{i,j,t}, s_{i,j,t-1}), \quad i = 1 \cdots n\\ & (y_{k,t} + \delta g_{C,k,t}, s_t) \leq \sum_{j \in \mathcal{J}} \theta_j (y_{k,j,t}, s_{k,j,t}), \quad k = 1 \cdots m\\ & \sum_{j=1}^{J} \theta_j = 1, \theta_j \geq 0, \delta \geq 0,\end{aligned} \tag{6.4}$$

If crop and soil are simultaneously expanded, we similarly obtain:

$$\begin{aligned} D^t(x_t, s_{t-1}, y_t, s_t; g_{C,t}) = & \max \delta \\ & (x_{i,t}, s_{i,t-1}) \geq \sum_{j \in \mathcal{J}} \theta_j (x_{i,j,t}, s_{i,j,t-1}), \quad i = 1 \cdots n \\ & (y_{k,t} + \delta g_{C,k,t}, s_{k,t}) \leq \sum_{j \in \mathcal{J}} \theta_j (y_{k,j,t}, s_{k,j,t}), \quad k = 1 \cdots m \\ & \sum_{j=1}^{J} \theta_j = 1, \theta_j \geq 0, \delta \geq 0, \end{aligned} \tag{6.5}$$

It follows that under a temporal separation of $\mathcal{T}$, one can compute $\mathcal{D}_T$, using Proposition 1. If there is no temporal separation, we need a specific linear program. Given to matrices $Z, Z' \in \mathbb{R}_+^{\tau \times p}$, we say that $Z \leq Z'$ if and only if $z_{i,t} \leq z'_{i,t}$, for all $i = 1, ..., p$ and all $t = 1, ..., \tau$. On can then define the overall technological process by:

$$\begin{aligned} \hat{\mathcal{T}} = \Big\{ (X, Y, S) \in \mathbb{R}_+^{\tau \times (n+m+q)} \colon (X, S_{-0}) \geq \sum_{j \in \mathcal{J}} \theta_j (X_j, S_{j,-0}), \\ (Y, S) \leq \sum_{j \in \mathcal{J}} \theta_j (Y_j, S_j), \sum_{j \in \mathcal{J}} \theta_j = 1, \theta \geq 0 \Big\}. \end{aligned} \tag{6.6}$$

It is then easy to deduce the following program:

$$\begin{aligned} \mathcal{D}_{T,\tau}(X, Y, S; G) = & \max \delta \\ & (x_{i,t}, s_{i,t-1}) \geq \sum_{j \in \mathcal{J}} \theta_j (x_{i,j,t}, s_{i,j,t-1}), \quad i = 1 \cdots n, t = 1 \cdots \tau \\ & (y_{k,t} + \delta g_{C,k,t}, s_{k,t}) \leq \sum_{j \in \mathcal{J}} \theta_j (y_{k,j,t}, s_{k,j,t}), k = 1 \cdots m, t = 1, ..., \tau - 1 \\ & (y_{k,\tau} + \delta g_{C,k,\tau}, s_{k,\tau} + \delta g_{S,k,\tau}) \leq \sum_{j \in \mathcal{J}} \theta_j (y_{k,j,\tau}, s_{k,j,\tau}), \quad k = 1 \cdots m, \\ & \sum_{j=1}^{J} \theta_j = 1, \theta_j \geq 0, \delta \geq 0, \end{aligned} \tag{6.7}$$

If there is no temporal separability, then we have the stric inequality:

$$\mathcal{D}_{T,\tau}(X, Y, S; G) - \min \Big\{ \min_{t=1,...,\tau-1} D_C^t((x_t, y_t, s_t, s_{t-1}; g_{C,t}), D^\tau(x_\tau, y_\tau, s_\tau, s_{\tau-1}; g_\tau) \Big\} < 0. \tag{6.8}$$

7 Conclusion

We have characterized the overall production process using a suitable distance function and different functional representations of the technology. We have shown how the technology can be modelled using linear programming methods.

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