Pollution-generating Technologies and Disposability Assumptions

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Abstract

This paper exams the concept of Pollution-generating Technologies (PgT). The main goal of this methodological contribution is to reveal any PgT in production processes compatible with a minimal set of assumptions. We model PgT using a congestion approach of the output set relaxing strong disposability assumption. To this end, we define the B-disposal assumption, that is a kind of limited strong disposability. The B-disposal assumption reflects cost disposability assumption with respect to the undesirable outputs. This disposability assumption leads to a new duality result between an output distance function and the revenue function with possibly negative shadow prices. A sample of 13 representative French airports is considered over the period 2007-2011, in order to implement the new B-disposal assumption on non-parametric technologies.

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Keywords: B-disposal Assumption, Bad Outputs, Cost Disposability, Distance Function, Duality, Pollution-generating Technologies (PgT), Revenue Function.

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1 Introduction

Since the early nineties, researchers strive to model undesirable outputs[1] using non-parametric models (Tyteca, 1996; Zhou et al., 2008a; Dakpo et al., 2016). Such methods need less restrictive assumptions than the econometric models (e.g. Murty and Kumar, 2003). Parametric methods involve to specify a functional form while non-parametric models require to apply mathematical programming methods; such as Data Envelopment Analysis (DEA). Furthermore, non-parametric approach presents more flexibility (e.g., in terms of inputs and outputs selection).

In general, several approaches are distinguished in the literature. Following Scheel (2001), the proposed models can be classified either into direct or indirect approaches. The former consider the original output data and alter the technology assumptions whereas the latter modify the value of the undesirable outputs.

The first approach was to treat bad outputs as inputs (Cropper and Oattes, 1992; Reinhard et al., 2000; Hailu and Veeman, 2001; Sahoo et al., 2011; Mahlberg et al., 2011). Färe and Grosskopf (2003) pointed out, through an illustrative example, that this method is inconsistent with physical laws. Following Pethig (2003, 2006), this approach also fails to satisfy the Materials Balance Principles (MBP)[2]. Moreover, considering residual outputs as inputs comes down to model the technology with an unbounded output set (Färe and Grosskopf, 2003; Leleu, 2013). Thus, this model fails to satisfy the standard axioms of production theory. Furthermore, it does not consider the link between the undesirable production and the inputs (Försund, 2009).

[1] Note that throughout this paper we use equivalently the terms bad outputs, undesirable outputs and residual outputs.

[2] More precisely, considering bad outputs as inputs fails to satisfy the first law of thermodynamics (Ayres and Kneese, 1969). This law can be illustrated through the famous saying of one of the founder of modern chemistry: "Nothing is lost, nothing is created, everything is transformed" Antoine Lavoisier (1743-1794).

The second approach attempts to model residual outputs in production theory by introducing additional production axioms. Färe et al. (1989) introduced model based on the concept of joint-production, using the Weak (or ray) Disposability (WD) axiom proposed by Shepard (1970), and the null jointness assumption. The former means that desirable and undesirable outputs can only be simultaneously reduced by a proportional factor. The latter highlights the pollution problem: any desirable production can be produced without bad outputs. Nevertheless, models derived from these notions have several limits. First, they consider a single abatement factor. Kuosmanen (2005) proposed to enhance them by introducing a non-uniform abatement factor in order to capture all feasible production plans. The traditional WD model, by considering a single abatement factor, reduces the production set and thereby conduct to an artificial high number of efficient Decision Making Units (DMUs). Second, the standard WD model does not exclude positive shadow prices for residual outputs (Hailu and Veeman, 2001; Hailu, 2003). Rödseth (2013) examines this issue, and finds that positive prices may be appropriate in cases where bads are recuperated by good outputs. Third, Kuosmanen and Podinovski (2009) pointed out that using a single abatement factor may yields some convexity infeasibilities. Finally, Coelli et al. (2007) showed that WD model fails to satisfy the MBP.

There exist also indirect approaches which alter the value of undesirable outputs in order to transform them into desirable outputs. Several authors consider an additive inverse transformation[3] (Koopmans, 1951), and the translation invariance property (Ali and Seidford, 1990; Seidford and Zhu, 2002), while other use a multiplicative inverse alteration (Golany and Roll, 1989). Then, standard DEA method can be implemented. However, as mentioned in Färe and Grosskopf (2004), such approaches is not consistent with physical laws since it

[3]The additive inverse transformation consists to multiply each undesirable outputs by -1. This approach exhibits the same technology set as considering bad outputs as inputs. However, it alters the sign of undesirable outputs.

consider strong disposal of outputs. Another issue is that it difficult to determine the suitable transformations of the bad outputs (Scheel, 2001).

Among the above approaches, WD models are extensively implemented in the literature about non-parametric combined environmental and productive efficiency studies. With respect to the limits associated with the WD model, two innovative approaches have been defined. First, an approach based on the MBP was introduced (Lauwers, and Van Huylenbroeck, 2003; Coelli et al., 2007; Lauwers, 2009). Most recently, Murty et al. (2012) proposed an innovative by-production technology constructed as an intersection of an intended-production technology and a residual-generation technology. Murty (2015) extended this approach to a full-blown axiomatic model. Dakpo et al. (2016) presented a critical review of these recent developments.

This paper proposes to model PgT using an innovative B-disposal assumption. This approach is based upon the congestion of the output set for with a relaxed disposability assumption is considered (Briec et al., 2016). We define a new B-disposal assumption that is a sort of limited strong disposability. B-disposal technologies allow to define congestion in the good outputs (i.e. loss of good outputs) resulting from the output set does not satisfy the usual disposal assumption. The B-disposal assumption reflects cost disposability assumption with respect to the undesirable outputs. Cost disposability involves that it is not possible to reduce freely bad outputs; i.e. without any costs. The main reason for this methodological innovation is to reveal any PgT in production processes compatible with a minimal set of assumptions. The basic tool employed to characterize multi-output technologies is the output distance function. Being dual to the revenue function (Shepard, 1953; Mc Fadden, 1978), it offers a general framework to economy analysis.

This note unfolds as follows. Section 2 presents the traditional technology, underlying standard axioms and their subsets. Furthermore, introduces the new disposal assumption and the boundaries for the residual outputs. Section 3 highlights the notions of output distance function and revenue function on the new PgT technology. Looking from a dual viewpoint, we establish the main duality result between the output distance function and a revenue function allowing for negative prices. Thereafter, we show how to detect cost disposability of undesirable outputs and testing consistency with revenue maximization. Section 4 defines convex non-parametric PgT and proposes a non-parametric test of cost disposability in bad outputs. A sample of 13 representative French airports is considered over the period 2007-2011, in order to implement the new B-disposal assumption on non-parametric technologies in section 5. Finally, Section 6 concludes, discusses limitations and offers directions for future research.

2 Technology: Assumptions and Definitions

2.1 Technology Based upon Traditional Assumptions

Let us define the notation used in this paper: $\mathbb{R}^n_+$ be the non-negative Euclidean n-dimensional orthant; for $y, \nu \in \mathbb{R}^n_+$ we denote $y \leq \nu \iff y_i \leq \nu_i \; \forall i \in [n]$, where $[n]$ denotes the subset $\{1, ..., n\}$.

A production technology transforming inputs $x = (x_1, ..., x_m) \in \mathbb{R}^m_+$ into outputs $y = (y_1, ..., y_n) \in \mathbb{R}^n_+$ can be characterized by the output correspondence $P : \mathbb{R}^m_+ \longrightarrow 2^{\mathbb{R}^n_+}$ where $P(x)$ is the set of all outputs vectors that can be produced from x:

$$P(x) = \{y : y \text{ can be produced from } x\}. \tag{2.1}$$

Throughout this paper, we assume the output correspondence satisfies the following regularity properties (see Hackman, 2008; Jacobsen, 1970; McFadden, 1978):

$P1$: $P(0) = \{0\}$ and $0 \in P(x)$ for all $x \in \mathbb{R}^m_+$.

$P2$: $P(x)$ is bounded above for all $x \in \mathbb{R}^m_+$.

$P3$: $P(x)$ is closed for all $x \in \mathbb{R}^m_+$.

Note that $P1$ imposes that there is no free lunch and that the null output can always be produced. Moreover, $P2$ and $P3$ involve that $P(x)$ is compact. In addition to the axioms of no free lunch as well as the boundedness and closedness of the output set, there are three other assumptions that we sometimes invoke on the output correspondence:

$P4$: $P(x)$ is a convex set for all $x \in \mathbb{R}^m_+$.

$P5$: If $u \geq x \Rightarrow P(x) \supseteq P(u)$.

$P6$: $\forall y \in P(x), 0 \leq v \leq y \Rightarrow v \in P(x)$.

Assumption $P4$ postulates convexity of the output correspondence. This is useful to provide a dual interpretation through the revenue function and in empirical applications of, for instance, non-parametric technologies. Notice that under $P1$ and $P4$ if $y \in P(x)$ then $\lambda y \in P(x)$, $\forall \lambda \in [0, 1]$. This implies the ray (or weak) disposability of the outputs, while axioms $P5$ and $P6$ imposes the more traditional assumption of strong (or free) disposal of inputs and outputs. A convex, ray disposable technology satisfying $P1 - P5$ but failing $P6$ is congested in the sense of Färe and Grosskopf (1983a).[4]

To measure efficiency, it is convenient to distinguish between certain subsets of the output set $P(x)$. In particular, two subsets denoting production units on the boundary prove useful. For all $x \in \mathbb{R}^m_+$, the efficient subset is defined by:

[4] Kuosmanen (2003) shows that this traditional specification fails convexity, but that a revised specification is convex.

$$E(x) = \{y \in P(x) : v \geq y \text{ and } v \neq y \Rightarrow v \notin P(x)\}. \tag{2.2}$$

The weak efficient subset is written as:

$$W(x) = \{y \in P(x) : v > y \Rightarrow v \notin P(x)\}. \tag{2.3}$$

2.2 Disposal Assumption for Bad Outputs

Let $B \subset [n]$, indexing the bad outputs of the technology. We introduce the following symbol:

$$y \geq^B v \iff \begin{cases} y_j \leq v_j & \text{if } j \in B \\ y_j \geq v_j & \text{else} \end{cases} \tag{2.4}$$

Moreover:

$$y >^B v \iff \begin{cases} y_j < v_j & \text{if } j \in B \\ y_j > v_j & \text{else} \end{cases} \tag{2.5}$$

Obviously, if $-y \geq^B -v$ we denote $y \leq^B v$. Notice that if $B = \emptyset$, then we retrieve the standard vector inequality, since the set of the residual outputs is empty.

We can now define a new disposability assumption for the outputs.

Definition 2.1 *Let P be an output correspondence satisfying P1-P3. For all $y \in \mathbb{R}^n_+$, the output set $P(x)$ satisfies the B-disposal assumption if for all sets of output vectors $\left\{y^J\right\}_{J \in \{\emptyset, B\}} \subset P(x)$, $y \leq^J y^J$ for any $J \in \{\emptyset, B\}$ implies that $y \in P(x)$.*

If $B = \emptyset$, then we retrieve B-disposal assumption reduces to the standard free disposability assumption.

In this paper, the free disposal assumption is limited by combining it with a particular partial reversion of free disposal. The more output dimensions are subjected to these particular partial reversions of free disposability defined by the B-disposal assumption, the more the traditional free disposability assumption gets limited and thus weakened. Indeed, Definition 2.1 implies that the larger the bad output subset B is the more difficult one can dispose outputs. In general, these definitions can account for cases where there is a simultaneous lack of free disposability in all dimensions, but it is also possible to define this lack independently in several dimensions.

Let us introduce the following convex cone:

$$K^B = \left\{ y \in \mathbb{R}^n : y \geq^B 0 \right\}. \tag{2.6}$$

Notice that this notation implies that $K^\emptyset = \mathbb{R}^n_+$. Definition 2.1 is illustrated in Figure 1. In the latter, we have $B = \{2\}$. For any y, if there is some $y^\varnothing$ that classically dominates y and some y^2 that "$\{2\}$-dominates" y, then $y \in P(x)$. For a given configuration of observations, this serves to construct an output set where wasting the second output (undesirable production) implies an additional opportunity revenue in terms of the first output dimension (desirable production). However, the reverse dependency between output dimensions does not hold. The B-disposal assumption reflects cost disposability assumption with respect to the bad outputs. Cost disposability implies that it is not possible to reduce freely residual outputs (y_2); i.e. without any costs.

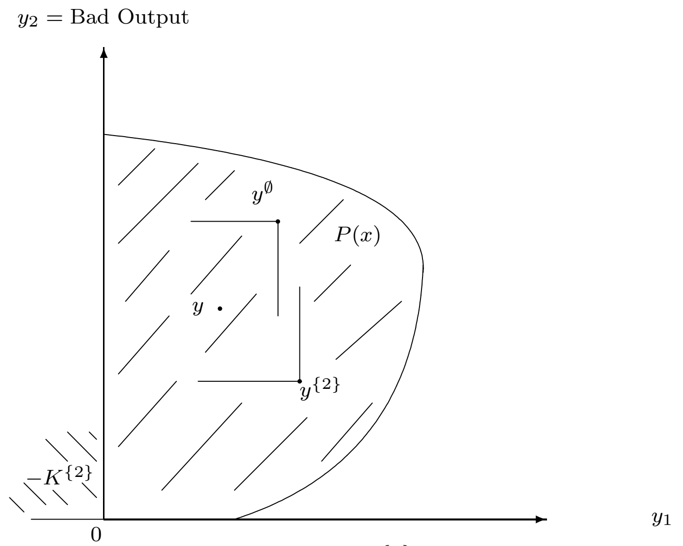

Figure 1: The case $B = \{2\}$ on an output set.

To study this new disposal assumption from a dual standpoint, we introduce the revenue function $R : \mathbb{R}^n \times \mathbb{R}^m_+ \longrightarrow \mathbb{R} \cup \{-\infty\}$ defined by:

$$R(p,x) = \begin{cases} \sup\limits_{y}\{p.y : y \in P(x)\} & \text{if } P(x) \neq \emptyset \\ -\infty & \text{if } P(x) = \emptyset \end{cases} \tag{2.7}$$

Notice that this definition allows to take into account negative prices which are specifically linked to PgT.

The following propositions study the properties of the B-disposal assumption.

Proposition 2.2 *Let P be an output correspondence satisfying P1-P3. For all $x \in \mathbb{R}^m_+$, $P(x)$ satisfies the B-disposal assumption if and only if:*

$$P(x) = \Big((P(x) - \mathbb{R}^n_+) \cap (P(x) - K^B)\Big) \cap \mathbb{R}^n_+.$$

This proposition characterizes a B-disposal output set in terms of an intersection of the convex cones in (2.6). Remark that 2.2 is only based on the B-disposal assumption and P1-P3. Therefore, the above proposition holds true even if $P(x)$ is not convex.

The following proposition extends the results of Proposition 2.2 to a convex output correspondence. In particular, we provide a dual characterization of the

B-disposability notion.

Proposition 2.3 *Let P be an output correspondence satisfying P1-P3. Moreover, assume that P4 holds. For all $x \in \mathbb{R}^m_+$, $P(x)$ satisfies the B-disposal assumption if and only if*

$$P(x) = \Big\{ y \in \mathbb{R}^n_+ : p.y \leq R(p, x), p \in \mathbb{R}^n_+ \cup K^B \Big\}.$$

Intuitively stated, a convex output set satisfying B-disposal can be enveloped by a revenue function for proper prices. This result constitutes the basis for the duality result developed in Section 3.

We are now ready to define a new cost disposability notion in the dimension of the residual outputs:

Definition 2.4 *Let P be an output correspondence satisfying P1-P3 and let B be a subset of $[n]$. For all $x \in \mathbb{R}^m_+$, $P(x)$ satisfies cost disposability of undesirable outputs if it fails strong disposability assumption but satisfies B-disposal assumption.*

In particular this means that:

$$(P(x) - \mathbb{R}^n_+) \cap \mathbb{R}^n_+ \neq \big((P(x) - \mathbb{R}^n_+) \cap (P(x) - K^B)\big) \cap \mathbb{R}^n_+. \tag{2.8}$$

Definition 2.4 provides a strict definition of cost disposability of bad outputs by assuming that the output set does not satisfy the usual disposal assumption. Recall that in such a case:

$$P(x) \neq (P(x) - \mathbb{R}^n_+) \cap \mathbb{R}^n_+. \tag{2.9}$$

In the following, for all price vector $p \in \mathbb{R}^n$, we say that an output of $P(x)$ is p-**optimal** if it maximizes the revenue $R(\cdot, p)$. An output vector $y \in P(x)$

is **interior**, if $y > 0$. The next result establishes a characterization of the new PgT.

Proposition 2.5 *Let P be an output correspondence that satisfies P1-P3. Assume that P4 holds. $P(x)$ satisfies cost disposability in the dimension of residual outputs if and only if there exists some interior p^B-optimal output in $P(x)$ with $p^B \in K^B \backslash \mathbb{R}^n_+$.*

2.3 Boundaries for Bad Outputs

It remains an open question: how to detect undesirable outputs from the structure of the output correspondence? To answer this question, it is useful to introduce the concept of bad frontier. Therefore, the following definition identifies a subset that is not efficient, but that is a part of the boundary of a B-disposal output correspondence.

Definition 2.6 *Let P be an output correspondence satisfying P1-P3 and let $B \subset [n]$. For all $x \in \mathbb{R}^m_+$, we call bad output efficient frontier the subset:*

$$E^B(x) = \{y \in P(x) : v \geq^B y \text{ and } v \neq y \Rightarrow v \notin P(x)\}.$$

We call bad output weakly efficient frontier the subset:

$$W^B(x) = \{y \in P(x) : v >^B y \Rightarrow v \notin P(x)\}.$$

It follows that $E^{\emptyset}(x) = E(x)$ is the usual efficient subset of $P(x)$. Moreover, note that $y \in E^B(x)$ if and only if:

$$(P(x) \backslash \{y\}) \cap (y + K^B) = \emptyset. \tag{2.10}$$

Proposition 2.7 *Let P be an output correspondence satisfying P1-P3. Assume that P4 holds.*
(a) The subsets $E^B(x)$ and $W^B(x)$ are closed.
(b) If the output set $P(x)$ satisfies cost disposability with respect to residual outputs then the subset $E^B(x)\backslash E(x)$ is non-empty and contains an interior point.
(c) Suppose that $E^B(x)\backslash E(x)$ is non-empty and contains an interior point. Suppose moreover that $P(x)$ satisfies the B-disposal assumption. Then $P(x)$ satisfies cost disposability in the dimension of undesirable outputs.

Remark 2.8 *There exist output sets that not satisfies cost disposability of bad outputs and for which there exists a boundary point in $E^B(x)\backslash E(x)$. For example assume that $P(x)$ is the cube defined by $P(x) = \{(y_1, y_2) \in \mathbb{R}^2_+ : y_1 \leq 1, y_2 \leq 1\}$. Then $y^B = (1, 0) \in E^{\{1\}}\backslash E$. However, $P(x)$ satisfies free disposability of undesirable outputs.*

3 Duality between Technology and Revenue Function Based on B-Disposability

Shephard (1953) introduced the so-called Shephard distance function in production theory. This distance function characterises technology and provides a useful tool in efficiency and productivity measurement by virtue of its radial nature[5]. This distance function has the advantage to be always feasible under P1-P4.

[5]See Russell (1985, 1987) for an axiomatic approach to the measurement of technical efficiency.

3.1 Distance Function and Revenue Function on PgT : A Duality Result

The output distance function $\psi_P : \mathbb{R}_+^{m+n} \longrightarrow \mathbb{R} \cup \{+\infty\}$ is defined by:

$$\psi_P(x, y) = \begin{cases} \inf\{\lambda > 0 : \frac{1}{\lambda} y \in P(x)\} & \text{if } \frac{1}{\lambda} y \in P(x) \text{ for some } \lambda > 0 \\ +\infty & \text{otherwise} \end{cases} \tag{3.1}$$

The above definition holds for a technology that satisfies the ray disposability assumption.

Following the traditional duality result in Jacobsen (1970) or McFadden (1978) between revenue function and output distance function, one can state a duality result making a link between the distance function and the revenue function on an output set $P(x)$ satisfying the ray disposability assumption.

Proposition 3.1 *Let P be an output correspondence satisfying P1-P5 and P6. We have the following properties:*
(a) For all $(x, y) \in \mathbb{R}_+^{m+n}$

$$\psi_P(x, y) = \inf_{p \geq 0} \Big\{ \frac{p.y}{R(p, x)} : R(p, x) \neq 0 \Big\}. \tag{3.2}$$

(b) Let p be a non-negative output price vector. We have:

$$R(p, x) = \sup_y \Big\{ \frac{p.y}{\psi(x, y)} : y \in \mathbb{R}_+^n \Big\}. \tag{3.3}$$

Apart from this traditional duality relationship, a weaker duality result between the revenue function and the ray (or weak) disposable output distance function is available in the literature (e.g. Shephard (1974)) whereby some (but not all) prices are allowed to be negative (assumption P6 is dropped).[6]

[6] Also McFadden (1978) anticipates the use of negative prices and maintains that duality

Now, we extend the properties of the distance function to account for negative orientations and to be compatible with output sets satisfying the B-disposal assumption.

Proposition 3.2 *Let P be an output correspondence satisfying P1-P5. Assume moreover that $P(x)$ satisfies the B-disposal assumption. We have the following properties:*
(a) For all $(x, y) \in \mathbb{R}_+^{m+n}$:

$$\psi_P(x, y) = \inf_{p \in K^B \cup \mathbb{R}_+^n} \big\{ \frac{p.y}{R(p, x)} : R(p, x) \neq 0 \big\}. \tag{3.4}$$

(b) Let $p \in K^B \cup \mathbb{R}_+^n$ be an output price vector having some negative components. Then:

$$R(p, x) = \sup_{y} \big\{ \frac{p.y}{\psi(x, y)} : y \in \mathbb{R}_+^n \big\}. \tag{3.5}$$

Property (a) extends the results by Shephard (1953) in the context of an output correspondence that may fail both the strong and the weak disposability assumptions. The converse results expressing the revenue function with respect to the Shephard distance function is stated in (b). This duality result considerably weakens current duality results imposing strong disposability. Otherwise stated, this proposition shows that B-disposal of outputs is a necessary and sufficient condition for the output Shephard distance function to characterize technology. This substantially weakens the existing result on the importance of ray disposal in the outputs for the traditional output distance function to characterize technology.

This new duality result is illustrated in Figure 2. Since the second output satisfies cost disposability, it receives a negative price and the revenue function ends up having a positive rather than a negative slope.

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results can be preserved under these circumstances.

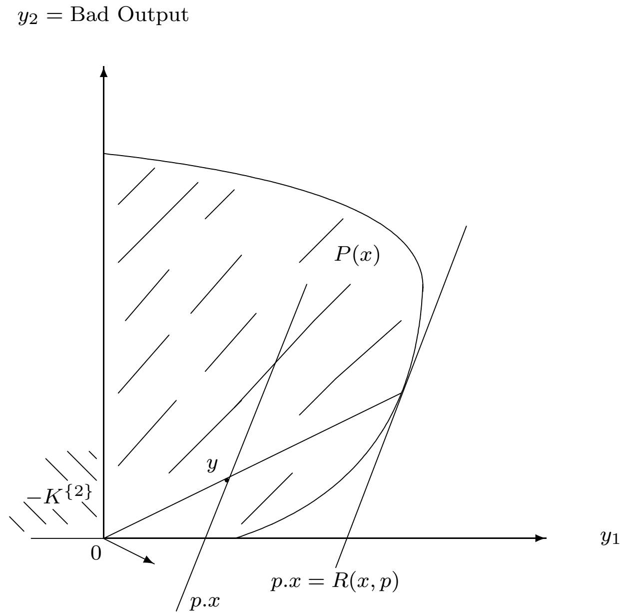

Figure 2: Shephard distance function and duality with $B = \{2\}$.

In principle it is possible to relax the convexity assumption. Under non-convexity, the duality result in Proposition 3.2 would only hold locally (similar to the local duality result in, e.g., Briec, Kerstens and Vanden Eeckaut (2004)). Note again that while the revenue function is non-decreasing in the outputs, revenue functions estimated on convex technologies are furthermore convex in the outputs (see Jacobsen (1970) or Shephard (1974)).

It should be clear by now that when the output set satisfies free disposal, then it also satisfies B-disposal assumption. But, the converse is not necessarily true. The same applies to weak disposal assumption: an output set satisfying weak disposability assumption also satisfies B-disposal assumption, but the converse need not be true.

3.2 Measurement of Cost Disposability

We are now interested in making the link between special cases of the output distance function introduced below and the cost disposability of bads. To study this relationship from the dual viewpoint we introduce the adjusted price correspondence $p : \mathbb{R}_+^{m+n} \longrightarrow 2^{\mathbb{R}^n}$ inspired from Luenberger (1995) and defined by:

$$p(x,y) = \arg\min_{p\in K^B\cup\mathbb{R}^n_+} \Big\{\frac{p.y}{R(p,x)} : R(p,x) \neq 0\Big\}. \tag{3.6}$$

Notice that if the minimum is not achieved, then $p(x,y)=\emptyset$. At points where $\psi_P(x,\cdot)$ is differentiable and applying the envelop theorem to 3.4 we obtain:

$$\nabla_y \psi_P(x,y) = \frac{p(x,y)}{R(p,x)}. \tag{3.7}$$

Thus,

$$p(x,y) = \nabla_y \psi_P(x,y) R(p,x) \tag{3.8}$$

For simplicity, we introduce the following notation:

$$P^{\emptyset}(x) = (P(x) - K^{\emptyset}) \cap \mathbb{R}^n_+ = (P(x) - \mathbb{R}^n_+) \cap \mathbb{R}^n_+, \tag{3.9}$$

$$P^B(x) = (P(x) - K^B) \cap \mathbb{R}^n_+, \tag{3.10}$$

$$P^J(x) = P^{\emptyset}(x) \cap P^B(x) = \Big((P(x) - \mathbb{R}^n_+) \cap (P(x) - K^B)\Big) \cap \mathbb{R}^n_+. \tag{3.11}$$

In the next proposition, the impact of adding convexity to axioms $P1-P3$ is analyzed.

Proposition 3.3 *Let P be an output correspondence satisfying P1-P4. For all $x \in \mathbb{R}^m_+$, we have the following properties:*
(a) $P(x)$ satisfies cost disposability with respect to residual outputs if and only if there exists some $y \in P(x)$ such that $p(x,y) \subset K^B \backslash \mathbb{R}^n_+$.
(b) $P(x)$ satisfies cost disposability in the dimension of undesirable outputs if and only if there exists some $y \in P(x)$ such that $\psi_{P^{\emptyset}}(x,y) < \psi_{P^J}(x,y)$.

In the following a procedure is proposed to measure cost disposability in the dimension of bads.

Definition 3.4 *Let P be an output correspondence satisfying P1-P3. For all production vector $(x, y) \in T$, we define the following ratio to measure cost disposability of residual outputs:*

$$DC^B(x, y) = \psi_{P^J}(x, y)/\psi_{P^\emptyset}(x, y)$$

.

We can now state the following corollary for our measure of cost disposability in the dimension of undesirable outputs.

Corollary 3.5 *Let P be an output correspondence satisfying P1-P3. Assume moreover that for all $x \in \mathbb{R}^m_+$, $P(x)$ satisfies the B-disposal assumption. Then, there exists some $y \in P(x)$ such that $DC^B(x, y) > 1$ if and only if $P(x)$ satisfies cost disposability with respect to bad outputs.*

This measure $DC^B(x, y)$ evaluates eventual cost disposability componentwise per subset B.

3.3 Testing for Consistency with Revenue Maximization

Suppose we are given some data on input-output vectors (x^j, y^j) and output prices p^j for all $j \in J$. Here we ask whether or not there exists a family of output sets $P(x)$ that can make sense of this observed behavior. It is possible to show that the existence of negative prices involves cost disposability of residual outputs in the general sense defined in this contribution. Following Varian (1984) we say that a family of output sets $P(x)$ rationalizes the data if y^j is a solution of the program:

$$\max_y \left\{p^j.y : y \in P(x^j)\right\} \tag{3.12}$$

for all $j \in J$. Equivalently, a family of output sets $P(x)$ rationalizes the data if for all $j \in J$ and all $y \in P(x^j)$:

$$p^j.y^j \geq p^j.y. \tag{3.13}$$

Assume that the output set is one-dimensional ($n = 1$). The main difference with Varian's (1984) Weak Axiom of Profit Maximization (WAPM) is that here prices can be negative. This excludes the strong disposal (or negative monotonic) property of the output set. Following Varian (1984) we assume the family of output sets is nested by the following assumption:

$$\forall y \in P(x),\ x \leq u \quad \text{implies that} \quad y \in P(u). \tag{3.14}$$

In the following, we suppose that for all j

$$p_i^j < 0 \text{ if } i \in B \quad \text{and} \quad p_i^j > 0 \text{ if } i \notin B \tag{3.15}$$

The key idea of the following result is that if an output set $P(x)$ rationalizes the data, then it necessarily satisfies a B-disposal assumption and cost disposability assumption in the undesirable outputs dimension.

Proposition 3.6 *The following conditions are equivalent:*
(a) There exists a family of nested output sets $P(x)$ that rationalizes the data.
(b) If $x^k \leq x^j$, then $p^j.y^k \leq p^j.y^j$ for all $j, k \in J$.
(c) There exists a family of nontrivial closed, convex and nested output sets that rationalizes the data and that satisfies cost disposability of bads.

An immediate consequence is that negative prices imply cost disposability in the dimension of residual outputs. Obviously, if all observed prices are non-negative, then we have $B = \emptyset$ for $j \in J$ and, because of $B = \emptyset$, we retrieve the

Varian (1984) WAPM result.

Notice that in principle it is possible to relax the convexity assumption (e.g., as in Briec, Kerstens and Vanden Eeckaut (2004)). Obviously, the same remarks as the ones mentioned at the end of subsection 3.1 apply.

4 Bad Outputs on Non-Parametric Technologies

In this section we focus on convex non-parametric technologies. In particular we consider the so-called Data Envelopment Analysis (DEA) model due to Banker, Charnes and Cooper (1984).

4.1 Non-Parametric Convex Technologies

We consider a set of DMUs $\mathcal{A} = \{(x_k, y_k) : k \in \mathcal{K}\}$ where $\mathcal{K}$ is an index set of natural number. We assume that the technology satisfy the Variable Returns to Scale (VRS) assumption (Banker et. al., 1984)[7]. In such case the production technology is defined by:

$$T^{DEA} = \Big\{(x, y) : x \geq \sum_{k \in \mathcal{K}} \mu_k x_k, y \leq \sum_{k \in \mathcal{K}} \mu_k y_k, \sum_{k \in \mathcal{K}} \mu_k = 1, \;\; \mu \geq 0\Big\} \qquad (4.1)$$

For any observed (x_0, y_0), the output correspondence is:

[7] Notice that if we assume that $\mathcal{A}$ contains the null input-output vector $(0, 0)$ then axiom P1 holds true. Equivalently, one can suppose a non-increasing returns to scale assumption (Färe et. al., 1983b).

$$P^{DEA}(x_0) = \Big\{ y: \ x_0 \geq \sum_{k \in \mathcal{K}} \mu_k x_k, y \leq \sum_{k \in \mathcal{K}} \mu_k y_k, \sum_{k \in \mathcal{K}} \mu_k = 1, \mu \geq 0 \Big\}$$

To establish cost disposability with respect to undesirable outputs, we need to identify the following subset:

$$P^{B,DEA}(x_0) = \Big\{ y : x_0 \geq \sum_{k \in \mathcal{K}} \theta_k x_k, y \leq^B \sum_{k \in \mathcal{K}} \theta_k y_k, \sum_{k \in \mathcal{K}} \theta_k = 1, \theta \geq 0 \Big\} \tag{4.2}$$

Let us consider the collection $J = \{\emptyset, B\}$. We now have $P^J(x_0) = P^{\emptyset}(x_0) \cap P^B(x_0) = \Big(\big(P(x_0) - \mathbb{R}^n_+ \big) \cap \big(P(x_0) - K^B \big) \Big) \cap \mathbb{R}^n_+$. Equivalently, we have:

$$P^{J,DEA}(x_0) = P^{DEA}(x_0) \cap P^{B,DEA}(x_0) \tag{4.3}$$

Thus, we have[8]

[8] As mentioned previously, one can define non-convex B-disposal technologies:

$$\begin{aligned} P^{J,DEA}_{NC}(x_0) = \Big\{ y : x_0 \geq \sum_{k \in \mathcal{K}} \theta_k x_k, \ & x_0 \geq \sum_{k \in \mathcal{K}} \mu_k x_k \\ y \leq^B \sum_{k \in \mathcal{K}} \theta_k y_k, \ & y \leq \sum_{k \in \mathcal{K}} \mu_k y_k \\ \theta, \mu \in \{0,1\} \Big\}. & \end{aligned} \tag{4.4}$$

$$
\begin{aligned}
P^{J,DEA}(x_0) = \Big\{ y : x_0 \geq \sum_{k \in \mathcal{K}} \theta_k x_k,\ & x_0 \geq \sum_{k \in \mathcal{K}} \mu_k x_k \\
y \leq^B \sum_{k \in \mathcal{K}} \theta_k y_k,\ & y \leq \sum_{k \in \mathcal{K}} \mu_k y_k \\
\sum_{k \in \mathcal{K}} \theta_k = \sum_{k \in \mathcal{K}} \mu_k = 1,\ & \theta, \mu \geq 0 \Big\}
\end{aligned}
$$

The above system of linear inequations can be formulated:

$$
\begin{aligned}
P^{J,DEA}(x_0) = \Big\{ y : x_{0,i} &\geq \sum_{k \in \mathcal{K}} \theta_k x_{k,i}, \ \ i = 1, ..., m \\
x_{0,i} &\geq \sum_{k \in \mathcal{K}} \mu_k x_{k,i}, \ \ i = 1, ..., m \\
y_j &\geq \sum_{k \in \mathcal{K}} \theta_k y_{k,j}, \ \ j \in B \\
y_j &\leq \sum_{k \in \mathcal{K}} \theta_k y_{k,j}, \ \ j = 1, ..., n \\
y_j &\leq \sum_{k \in \mathcal{K}} \mu_k y_{k,j}, \ \ j = 1, ..., n \\
\sum_{k \in \mathcal{K}} \theta_k &= \sum_{k \in \mathcal{K}} \mu_k = 1, \ \ \theta, \mu \geq 0 \Big\}
\end{aligned}
\tag{4.5}
$$

Remark that, if $\theta = \mu$ then the above output set shows non-disposabilty of undesirable outputs (Kuosmanen, 2005). Following Leleu (2013), such representation is an incorrect modeling of VRS assumption in traditional Shepard's weakly disposable technology. Nevertheless, this modeling has been implemented in the literature (see for instance Picazo-Tadeo et al., 2005; Bilsel et al., 2014). This contribution provides an innovative axiomatic characterization of the incorrect modeling of VRS assumption in traditional Shepard's weakly disposable technology.

Notice that if we consider a set of DMUs $\mathcal{A}^{'} = \{(x_k, y_k), (x_k, 0) : k \in \mathcal{K}\}$[9], where $\mathcal{K}$ is an index set of natural number, we retrieve the correct way of linearizing VRS Shepard's weakly disposable technology proposed in Kuosmanen (2005). Kuosmanen and Podinovski (2009), showed that this technology is the smallest convex extension of Shepard's weakly disposable technology. Based on the initial work of Podinovski (2004), they highlighted that Kuosmanen technology is the correct minimum extrapolation technology that verified the stated axioms. This modeling allows to consider proper abatement factor for each observed activity. Furthermore, based on this modeling, a dual interpretation of weak disposability is proposed in Kuosmanen and Matin (2011)[10]. This paper offers a new axiomatic characterization of the Kuosmanen technology.

In the same vein, if we take $\mathcal{A}_0^{'} = \{(x_k, y_k), (x_0, 0) : k \in \mathcal{K}\}$[8], the above output set corresponds to the correct way of linearizing VRS Shepard's weakly disposable technology proposed in Leleu (2013). This modeling also introduce a dual interpretation of weak disposability.

[9] In such a case axiom P1 holds true.

[10] Kuosmanen and Matin (2011) introduced the concept of "limited liability condition" to provide a dual interpretation of the weak disposability. If the maximum profit is not positive and smaller than the sunk costs of inputs the "limited liability condition" is not verified. In such a case, it is optimal to stop the production activity.

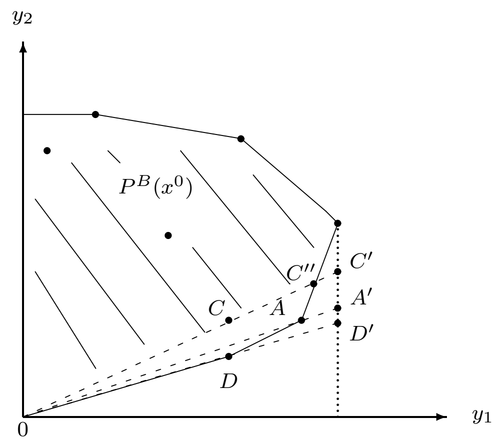

Figure 3: A non parametric Test with $B = \{2\}$.

In the figure 3, cost disposability with respect to residual outputs can be detected at points A, C and D.

We can now state the following result:

Proposition 4.1 *The non-parametric convex output correspondence satisfies the following properties.*
(a) $P^{J,DEA}$ *is convex;*
(b) $P^{J,DEA}$ *satisfies the* B*-disposal assumption;*
(c) $P^{J,DEA}$ *is a closed subset of* $\mathbb{R}^n_+$.

4.2 By-production technology and generalised B-disposal assumption

Murty et al. (2012) proposed an innovative by-production technology constructed as an intersection of an intended-production technology and a residual-generation technology. The definition of a new generalised B-disposal assumption allows to introduce a similar approach of the Murty et al.'s (2012) method.

We first define the notation used to define a generalised version of the B-disposal assumption. Let $B = \{B_{\text{in}}, B_{\text{out}}\} \subset [m] \times [n]$, indexing the inputs generating pollution and the bad outputs of the technology. Let T a production technology satisfying the following regularity properties:

$T1$: $(0,0) \in T$ and $(0,y) \in T \Rightarrow y = 0$.

$T2$: $T(y) = \{(u,v) \in T : v \leq y\}$ is bounded for all $y \in \mathbb{R}^n_+$.

$T3$: T is closed.

$T4$: $\forall (x,y) \in T \ \ \wedge \ \ \forall (u,v) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+$ if $(x,-y) \leq (u,-v)$ then $(u,v) \in T$.

The assumptions $T1 - T3$ are equivalent to $P1 - P3$. T4 imposes traditional assumption of strong disposability of inputs and outputs.

Definition 4.2 *Let T a production technology satisfying T1-T3. For all $(x,y) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+$, the technology T satisfies the generalised B-disposal assumption if for all sets of vectors $\left\{x^J, y^J\right\}_{J\in\{\emptyset,B\}} \subset T$, $(-x,y) \leq^J (-x^J, y^J)$ for any $J \in \{\emptyset, B\}$ implies that $(x,y) \in T$.*

If $B = \emptyset$, then the generalised B-disposal assumption reduces to the standard free disposability assumption ($T4$).

Proposition 4.3 *Let T a technology satisfying T1-T3. For all $(x,y) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+$, T satisfies the generalised B-disposal assumption if and only if:*

$$T = \Big(\big(T + (\mathbb{R}^m_+ \times (-\mathbb{R}^n_+))\big) \cap \big(T + (K^{B_{in}} \times (-K^{B_{out}}))\big)\Big) \cap (\mathbb{R}^m_+ \times \mathbb{R}^n_+).$$

For simplicity, we introduce the following notation:

$$T^{\emptyset} = \big(T + (\mathbb{R}^m_+ \times (-\mathbb{R}^n_+))\big) \cap (\mathbb{R}^m_+ \times \mathbb{R}^n_+), \tag{4.6}$$

$$T^{B} = \big(T + (K^{B_{\text{in}}} \times (-K^{B_{\text{out}}}))\big) \cap (\mathbb{R}^m_+ \times \mathbb{R}^n_+), \tag{4.7}$$

$$T^J = T^{\emptyset} \cap T^B = \Big(\big(T + (\mathbb{R}^m_+ \times (-\mathbb{R}^n_+))\big) \cap \big(T + (K^{B_{\text{in}}} \times (-K^{B_{\text{out}}}))\big) \Big) \cap (\mathbb{R}^m_+ \times \mathbb{R}^n_+). \tag{4.8}$$

We assume that the technology satisfy the Variable Returns to Scale (VRS) assumption (Banker et. al., 1984). To establish generalised cost disposability with respect to polluting inputs and undesirable outputs, we need to identify the following subset:

$$T^{B,DEA} = \Big\{ (x,y) : x \geq^{B_{\text{in}}} \sum_{k \in \mathcal{K}} \theta_k x_k, y \leq^{B_{\text{out}}} \sum_{k \in \mathcal{K}} \theta_k y_k, \sum_{k \in \mathcal{K}} \theta_k = 1, \ \ \theta \geq 0 \Big\} \tag{4.9}$$

Let us consider the collection $J = \{\emptyset, B\}$. We now have $T^J = T^{\emptyset} \cap T^B = \Big(\big(T + (\mathbb{R}^m_+ \times (-\mathbb{R}^n_+))\big) \cap \big(T + (K^{B_{\text{in}}} \times (-K^{B_{\text{out}}}))\big) \Big) \cap (\mathbb{R}^m_+ \times \mathbb{R}^n_+)$. Equivalently, we have:

$$T^{J,DEA} = T^{DEA} \cap T^{B,DEA} \tag{4.10}$$

Thus, we have

$$\begin{aligned} T^{J,DEA} = \Big\{ (x,y) : x \geq \sum_{k \in \mathcal{K}} \mu_k x_k, x \geq^{B_{\text{in}}} \sum_{k \in \mathcal{K}} \theta_k x_k \\ y \leq \sum_{k \in \mathcal{K}} \mu_k y_k, y \leq^{B_{\text{out}}} \sum_{k \in \mathcal{K}} \theta_k y_k \\ \sum_{k \in \mathcal{K}} \theta_k = \sum_{k \in \mathcal{K}} \mu_k = 1, \ \ \theta, \mu \geq 0 \Big\} \end{aligned}$$

The subset $T^{B,DEA}$ allows to capture cost disposability in the dimensions of inputs generating pollution and residual outputs. In the Murty et al.'s (2012) words, this sub-technology reflects nature's residual generation. The subset T^{DEA} permits to capture the intended-production activities of firms. The intersection of $T^{B,DEA}$ and T^{DEA} defines a new PgT. Murty et al. (2012) assume that the nature's residual generation sub-technology operates independently of the firm's intended-production sub-technology. The proposed PgT no postulates a such assumption. The subset $T^{B,DEA}$ is dependent on the intended (desirable) outputs and on no polluting inputs; i.e., inequalities need to be specified for these inputs and outputs in 4.9. We not assume that intended outputs and no-polluting inputs not interact with the nature's residual generation sub-technology. The above system of linear inequations can be rewritten as follows:

$$
\begin{aligned}
T^{J,DEA} = \Big\{(x,y) : x_i \le \sum_{k\in\mathcal{K}} \theta_k x_{k,i}, \;\; & i \in B_{\text{in}} \\
x_i \ge \sum_{k\in\mathcal{K}} \theta_k x_{k,i}, \;\; & i = 1, ..., m \\
x_i \ge \sum_{k\in\mathcal{K}} \mu_k x_{k,i}, \;\; & i = 1, ..., m \\
y_j \ge \sum_{k\in\mathcal{K}} \theta_k y_{k,j}, \;\; & j \in B_{\text{out}} \\
y_j \le \sum_{k\in\mathcal{K}} \theta_k y_{k,j}, \;\; & j = 1, ..., n \\
y_j \le \sum_{k\in\mathcal{K}} \mu_k y_{k,j}, \;\; & j = 1, ..., n \\
\sum_{k\in\mathcal{K}} \mu_k = 1, \sum_{k\in\mathcal{K}} \theta_k = 1, \;\; & \mu \ge 0, \;\; \theta \ge 0\Big\}
\end{aligned}
\tag{4.11}
$$

Note that the above PgT not consider abatement outputs, but obviously it is easy to introduce such outputs. We just have to insert the following constraint:

$y_j \geq \sum_{k\in\mathcal{K}} \mu_k y_{k,j}, \;\; j \in B'_{\text{out}}$. Where, $B = \{B_{\text{in}}, B_{\text{out}}, B'_{\text{out}}\} \subset [m] \times [n]$ indexing the inputs generating pollution, the bad outputs and the abatement outputs of the technology. Finally, remark that adding the following constraints[11] in 4.11:

$$\sum_{k\in\mathcal{K}} \theta_k x_{k,i} = \sum_{k\in\mathcal{K}} \mu_k x_{k,i}, \;\; i = 1, ..., m$$

and

$$\sum_{k\in\mathcal{K}} \theta_k y_{k,j} = \sum_{k\in\mathcal{K}} \mu_k y_{k,j}, \;\; j = 1, ..., n.$$

Then, the PgT defined in 4.11 can be rewritten in the by-production technology (Murty et al., 2012).

4.3 Non-Parametric Test of Cost Disposability in the Dimension of Bads

To test cost disposability with respect to undesirable outputs we need to be able to compute the distance function over an output correspondence. From the specification of convex non-parametric technologies, it is quite straightforward to derive the following mathematical program[12]:

[11]These additional constraints assume that the efficient combination of the inputs and outputs should be the same in both sub-technologies.

[12]Remark that, if $\theta = \mu$ then $\psi_{PJ,DEA}(x_0, y_0)$ can be implemented based on the set of DMUs $\mathcal{A}$, $\mathcal{A}^{'}$ or $\mathcal{A}_0^{'}$.

$$
\begin{aligned}
\psi_{PJ,DEA}(x_0, y_0) = \inf \lambda & \\
s.t.\ \ x_{0,i} &\geq \sum_{k \in \mathcal{K}} \theta_k x_{k,i}, \ \ i = 1, ..., m \\
x_{0,i} &\geq \sum_{k \in \mathcal{K}} \mu_k x_{k,i}, \ \ i = 1, ..., m \\
\frac{1}{\lambda} y_{0,j} &\geq \sum_{k \in \mathcal{K}} \theta_k y_{k,j}, \ \ j \in B \\
\frac{1}{\lambda} y_{0,j} &\leq \sum_{k \in \mathcal{K}} \theta_k y_{k,j}, \ \ j = 1, ..., n \\
\frac{1}{\lambda} y_{0,j} &\leq \sum_{k \in \mathcal{K}} \mu_k y_{k,j}, \ \ j = 1, ..., n \\
\sum_{k \in \mathcal{K}} \theta_k &= \sum_{k \in \mathcal{K}} \mu_k = 1, \ \ \theta, \mu \geq 0
\end{aligned}
$$

The above program has $2(m+n)+1+\text{Card}(B)$ constraints, where $\text{Card}(B)$ is the number of elements in B. When the technology is DEA convex, then the solution is obtained by solving a linear program. To measure cost disposability of residual outputs we need to compute $\psi_{PJ,DEA}(x^0, y^0)/\psi_{PDEA}(x^0, y^0)$[13]. In the same way $\psi_{PDEA}(x_0, y_0)$ can be computed as follows:

[13]Consider replacing the VRS DEA technologies by CRS technologies and that $\theta = \mu$, then the test of cost disposability with respect to undesirable outputs is equivalent of the test of congestion in Färe et al. (1989) (not paying attention to the choice of distance function).

$$
\begin{aligned}
\psi_{PDEA}(x_0, y_0) = \inf \lambda & \\
& s.t.\ \ x_{0,i} \geq \sum_{k\in\mathcal{K}} \theta_k x_{k,i}, \ \ i = 1, ..., m \\
& \quad \frac{1}{\lambda} y_{0,j} \leq \sum_{k\in\mathcal{K}} \theta_k y_{k,j}, \ \ j = 1, ..., n \\
& \quad \sum_{k\in\mathcal{K}} \theta_k = 1, \ \ \theta \geq 0
\end{aligned}
$$

5 Empirical illustration

5.1 Data

The dataset used comes from many reports and documents of the Ministère de l'écologie, du Développement durable et de l'Énergie (http://www.developpement-durable.gouv.fr). Two inputs are selected: (i) number of employees and (ii) operational costs. These inputs indicators permit to produce different outputs. Thus, one desirable output, (iii) number of passengers ; and one undesirable output represented by (iv) CO_2 emissions. This bad output is measured by using the TARMAAC (Traitements et Analyses des Rejets éMis dans l'Atmosphère par l'Aviation Civile) tool of the Direction générale de l'Aviation civile (DGAC).

Table 1 presents the statistic descriptives of the variables used in this study.

5.2 Results

Table 2 presents measure of cost disposability in the dimension of residual outputs based on the B-disposability (DC^B) and the weak disposability (DC^{WD})

Table 1: Characteristics of inputs and outputs

Variables	Min	Max	Mean	St. Dev.
	Inputs			
Employees (quantity)	67	3813	738	1166
Operational costs (Keuros)	15614	1112248	187521	329679
	Good Output			
Passengers (quantity)	1014704	60970551	10328725	15646444
	Bad Output			
CO_2 emissions (millions of tons)	13	896	136	222

assumptions for the years 2007 to 2011.

Notice that both weak disposable and B-disposable Shepard output distance functions are computed under VRS assumption (columns three and four). We consider the correct linearization of the weakly disposable technology proposed by Sahoo et al. (2011) or Zhou et al. (2008b):

$$
\begin{aligned}
\psi_{PWD,DEA}(x_0, y_0) &= \inf \lambda \\
& s.t.\ \ x_{0,i} \geq \sum_{k \in \mathcal{K}} \alpha_k x_{k,i}, \ \ i = 1, ..., m \\
& \quad \frac{1}{\lambda} y_{0,j} = \sum_{k \in \mathcal{K}} \alpha_k y_{k,j}, \ \ j \in B \\
& \quad \frac{1}{\lambda} y_{0,j} \leq \sum_{k \in \mathcal{K}} \alpha_k y_{k,j}, \ \ j = card(B) + 1, ..., n \\
& \quad \sum_{k \in \mathcal{K}} \alpha_k = \theta, \ \ \alpha \geq 0, \ \ 0 \leq \theta \leq 1
\end{aligned}
$$

Readers can see that the B-disposability assumption allows to consider a more severe form of cost disposability with respect to undesirable outputs than it proposed by the weak disposability assumption. For each time period we have: Shep. BD $\geq$ Shep. WD or equivalently $DC^B \geq DC^{WD}$.

Table 2: Measures of congestion in good outputs

Airport	Shep. SD	Shep. WD	Shep. BD	DC^{WD}	DC^{B}
2007					
Bâle-Mulhouse	0.7900	0.7900	0.7900	1.0000	1.0000
Beauvais	0.6569	1.0000	1.0000	1.5224	1.5224
Bordeaux-Mérignac	1.0000	1.0000	1.0000	1.0000	1.0000
Lille	0.6419	0.6419	1.0000	1.0000	1.5578
Lyon-Saint Exupéry	0.9091	0.9091	0.9091	1.0000	1.0000
Marseille-Provence	0.9924	1.0000	1.0000	1.0077	1.0077
Montpellier-Méditerranée	0.7503	0.8927	1.0000	1.1898	1.3328
Nantes-Atlantique	0.8713	0.8713	0.8811	1.0000	1.0113
Nice-Côte d'azur	1.0000	1.0000	1.0000	1.0000	1.0000
Paris CDG	1.0000	1.0000	1.0000	1.0000	1.0000
Paris ORY	0.5604	1.0000	1.0000	1.7844	1.7844
Strasbourg-Entzheim	1.0000	1.0000	1.0000	1.0000	1.0000
Toulouse-Blagnac	1.0000	1.0000	1.0000	1.0000	1.0000
2008					
Bâle-Mulhouse	0.7956	0.7956	0.7956	1.0000	1.0000
Beauvais	0.6212	1.0000	1.0000	1.6098	1.6098
Bordeaux-Mérignac	1.0000	1.0000	1.0000	1.0000	1.0000
Lille	0.6944	0.6944	1.0000	1.0000	1.4400
Lyon-Saint Exupéry	0.9960	0.9960	0.9960	1.0000	1.0000
Marseille-Provence	0.9353	0.9353	0.9353	1.0000	1.0000
Montpellier-Méditerranée	0.7354	0.8793	1.0000	1.1957	1.3598
Nantes-Atlantique	0.9433	0.9433	0.9788	1.0000	1.0376
Nice-Côte d'azur	1.0000	1.0000	1.0000	1.0000	1.0000
Paris CDG	1.0000	1.0000	1.0000	1.0000	1.0000
Paris ORY	0.5474	1.0000	1.0000	1.8269	1.8269
Strasbourg-Entzheim	0.8689	0.8689	1.0000	1.0000	1.1509
Toulouse-Blagnac	1.0000	1.0000	1.0000	1.0000	1.0000
2009					
Bâle-Mulhouse	0.8410	0.8410	0.8410	1.0000	1.0000
Beauvais	0.8037	1.0000	1.0000	1.2442	1.2442
Bordeaux-Mérignac	1.0000	1.0000	1.0000	1.0000	1.0000
Lille	0.7731	0.7731	1.0000	1.0000	1.2935
Lyon-Saint Exupéry	0.9504	0.9504	0.9504	1.0000	1.0000
Marseille-Provence	1.0000	1.0000	1.0000	1.0000	1.0000
Montpellier-Méditerranée	0.7548	0.7980	1.0000	1.0573	1.3249
Nantes-Atlantique	0.9987	0.9987	1.0000	1.0000	1.0013
Nice-Côte d'azur	1.0000	1.0000	1.0000	1.0000	1.0000
Paris CDG	1.0000	1.0000	1.0000	1.0000	1.0000
Paris ORY	0.5470	1.0000	1.0000	1.8281	1.8281
Strasbourg-Entzheim	0.8738	0.8738	1.0000	1.0000	1.1444
Toulouse-Blagnac	1.0000	1.0000	1.0000	1.0000	1.0000
2010					
Bâle-Mulhouse	0.8384	0.8384	0.8384	1.0000	1.0000
Beauvais	0.7981	1.0000	1.0000	1.2530	1.2530
Bordeaux-Mérignac	1.0000	1.0000	1.0000	1.0000	1.0000
Lille	0.7758	0.8812	1.0000	1.1358	1.2889
Lyon-Saint Exupéry	0.9219	0.9252	0.9252	1.0036	1.0036
Marseille-Provence	1.0000	1.0000	1.0000	1.0000	1.0000
Montpellier-Méditerranée	0.7200	0.7597	1.0000	1.0552	1.3890
Nantes-Atlantique	0.9497	0.9497	0.9649	1.0000	1.0160
Nice-Côte d'azur	1.0000	1.0000	1.0000	1.0000	1.0000
Paris CDG	1.0000	1.0000	1.0000	1.0000	1.0000
Paris ORY	0.5480	1.0000	1.0000	1.8249	1.8249
Strasbourg-Entzheim	0.8097	0.8097	1.0000	1.0000	1.2350
Toulouse-Blagnac	0.9793	0.9793	0.9880	1.0000	1.0089
2011					
Bâle-Mulhouse	0.8671	0.9147	0.9147	1.0549	1.0549
Beauvais	0.8006	1.0000	1.0000	1.2490	1.2490
Bordeaux-Mérignac	1.0000	1.0000	1.0000	1.0000	1.0000
Lille	0.7034	0.7707	1.0000	1.0957	1.4216
Lyon-Saint Exupéry	0.9593	1.0000	1.0000	1.0424	1.0424
Marseille-Provence	1.0000	1.0000	1.0000	1.0000	1.0000
Montpellier-Méditerranée	0.6885	0.7151	1.0000	1.0387	1.4525
Nantes-Atlantique	0.9300	0.9300	0.9561	1.0000	1.0282
Nice-Côte d'azur	1.0000	1.0000	1.0000	1.0000	1.0000
Paris CDG	1.0000	1.0000	1.0000	1.0000	1.0000
Paris ORY	0.5629	1.0000	1.0000	1.7766	1.7766
Strasbourg-Entzheim	0.7448	0.7448	1.0000	1.0000	1.3427
Toulouse-Blagnac	1.0000	1.0000	1.0000	1.0000	1.0000

6 Conclusions

This paper is aimed to analyze the concept of PgT in the context of multi-output production and duality theory. In this contribution a class of PgT based upon a new notion of B-disposal assumption is considered. This new B-disposal assumption consists to re-interpret the traditional strong disposability assumption as a limited rather than a global property.

Thereafter, we introduce a new duality result between the output distance function and the revenue function with possibly negative shadow prices. This new duality result substantially weakens the existing result on the importance of weak disposal in the outputs for the traditional output distance function to characterize technology.

Then, introducing non parametric convex PgT, this contribution provides an innovative axiomatic characterization of the incorrect modeling of VRS assumption in traditional Shepard's weakly disposable technology. Under specific sets of DMUs (respectively $\mathcal{A}'$ and $\mathcal{A}'_0$), we retrieve the correct way of linearizing VRS Shepard's weakly disposable technology proposed in Kuosmanen (2005) and Leleu (2013). Moreover, we show that the new PgT can be rewritten in the by-production technology (Murty et al., 2012).

Finally, we propose an empirical illustration to emphasize the new measure of cost disposability in the dimension of bad outputs (DC^B). We show that B-disposability assumption defines a more severe form of cost disposability with respect to residual outputs than it suggested by the weak disposability assumption.

The principal limitation of this paper relates of one main reason. We focus on the output distance function and it dual relation with the revenue function. The duality result and the new measure of cost disposability of bad outputs can also defined using the so called directional distance function (Luenberger, 1992, 1995; Chambers et. al., 1996; Chambers and Pope, 1996).

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Appendix:

Proof of Proposition 2.2: First, assume that $P(x)$ satisfies the B-disposal assumption. For all sets of output vectors $\left\{y^J\right\}_{J\in\{\emptyset,B\}} \subset P(x)$ and all $y \in \mathbb{R}^n_+$, $y \leq^J y^J$ $\forall J \in \{\emptyset, B\}$ implies that $y \in P(x)$. Consequently, we deduce that $(\bigcap_{J\in\{\emptyset,B\}} P(x) - K^J) \bigcap \mathbb{R}^n_+ \subset P(x)$. Moreover, we obviously have $P(x) \subset (\bigcap_{J\in\{\emptyset,B\}} P(x) - K^J) \bigcap \mathbb{R}^n_+$. Hence, the first implication holds. Conversely, assume that $P(x) = (\bigcap_{J\in\{\emptyset,B\}} P(x) - K^J) \bigcap \mathbb{R}^n_+$. For any $J \in \{\emptyset, B\}$, if $y^J \in P(x)$ and $y \leq^J y^J$, then $y \in (P(x) - K^J) \bigcap \mathbb{R}^n_+$. Consequently, let a set of output vectors $\left\{y^J\right\}_{J\in\{\emptyset,B\}} \subset P(x)$, $y \in (P(x) - K^J) \bigcap \mathbb{R}^n_+ = P(x)$. Thus, $P(x)$ satisfies the B-disposal assumption. □

Proof of Proposition 2.3: We have $P(x) = (\bigcap_{J\in\{\emptyset,B\}} P(x) - K^J) \bigcap \mathbb{R}^n_+$. But, for any $J \in \{\emptyset, B\}$ (since P4 holds),

$$P(x) - K^J = \bigcap_{p\in K^J} \left\{y \in \mathbb{R}^n_+ : p.y \leq R(p, x)\right\}$$

Consequently,

$$P(x) = \Big(\bigcap_{J\in\{\emptyset,B\}} \bigcap_{p\in K^J} \{y \in \mathbb{R}^n_+ : p.y \leq R(p, x)\}\Big) \cap \mathbb{R}^n_+.$$

This subset can immediately be rewritten as:

$$P(x) = \Big\{y \in \mathbb{R}^n_+ : p.y \leq R(p, x), p \in \bigcup_{J\in\{\emptyset,B\}} K^J\Big\}.$$

By using Proposition 2.2, this ends the proof. □

Proof of Proposition 2.5: We first prove that if there exists a price $p \in K^B \backslash \mathbb{R}^n_+$ and an interior p^B-optimal output vector, then the technology satisfies cost disposability in the dimension of bad outputs. Suppose that $p \in K^B \backslash \mathbb{R}^n_+$. In such a case, there is some $j \in [n] \backslash B$ such that $p_j < 0$. Assume that the free disposal assumption holds and let us show a contradiction. Since $P(x)$ is a compact subset of $\mathbb{R}^n_+$, there exists some y^B maximizing the revenue. Since $p_j < 0$ and under a free disposal assumption, at the optimum, we have $y^B_j = 0$. Therefore, y^B is not an interior point. Consequently, if there exists a p^B-optimal interior point, the output set is not freely disposable. Thus the output set $P(x)$ satisfies cost disposability with respect to undesirable outputs.

Let us prove the converse. Suppose that the output set satisfies cost disposability in the dimension of residual outputs. In such a case there exists some frontier point $z^B > 0$ that belongs to $P(x)$ and some price vector $r^B \in K^B \backslash \mathbb{R}^n_+$ such that z^B maximizes the revenue for price r^B. In the following we prove that in the case where $r^B \in \mathbb{R}^n_+$ one can find a pair (y^B, p^B) such that that y^B is an interior p^B-optimal point with $p^B \in K^B \backslash \mathbb{R}^n_+$. Since $P(x)$ is closed, convex and contains 0 and using the fact that $z^B > 0$, it is easy to show that one can find some frontier output vector $y^B > 0$ in $P(x)$. Moreover, from the hyperplane support theorem there exists some price vector $p^B \in \mathbb{R}^n$ with $p^B.y^B = R(p^B, x)$. Clearly $p^B \notin \mathbb{R}^n_+$ because in such case we would have $p^B.y^\star > p^B.y^B$, which contradicts the fact that y^B maximizes the revenue.

We have proven that one can find a pair (y^B, p^B) such that $y^B > 0$ is p^B-optimal with $p^B \notin \mathbb{R}^n_+$. Hence, all we need to prove is that $p^B \in K^B$. Suppose this is not the case end let us show a contradiction. If this is not true, there is some $j \in B$ with $p^B_j < 0$. Let us consider the output vector $\tilde{y}^B$ defined for all $i \in [n]$ by

$$\tilde{y}^B_i = \begin{cases} y^B_i & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

We obviously have $\tilde{y}^B \leq^B y^B$. Since the B-disposal assumption holds $\tilde{y}^B \in P(x)$.

However, since $p_j^B < 0$, we have $p^B.\tilde{y}^B > p^B.y^B$ that is a contradiction. Therefore, $p_i^B \geq 0$ for all $i \in B$ which implies that $p^B \in K^B$ and ends the proof. $\square$

Proof of Proposition 2.7: (a) is a standard result whose the proof similar to that one of the standard case. (b) Suppose that the technology satisfies cost disposability with respect to bad outputs. From Proposition 2.5 there is an interior output vector $y^B \in P(x)$ and a price $p^B \in K^B$ such that y^B maximizes the revenue in $P(x)$. Since $p^B \in K^B$, the hyperplane $\{y : p^B \cdot y = R(x, p^B)\}$ weakly separates $P(x)$ and $y^B + K^B$. It follows that $(P(x)\backslash\{y^B\}) \cap (y^B + K^B) = \emptyset$. Consequently, $y^B \in E^B(x)$ which proves (b). To prove (c) suppose that there is some $z^B \in E^B(x)$ in the interior of $P(x)$. Equivalently, $(P(x)\backslash\{z^B\}) \cap (z^B + K^B) = \emptyset$. From the convex separation theorem there is some price vector r^B such that the hyperplane $\{z : r^B.z = R(x, r^B)\}$ weakly separates $P(x)$ and $z^B + K^B$. However, from the Farkas lemma, this property implies that $r^B \in K^B$. If $r^B \notin \mathbb{R}^n_+$, then from Proposition 2.5, the result is established. If this is not the case, since the output set is closed convex and contains 0, following the method used in the proof of Proposition 2.5 one can find a frontier point $y^B > 0$ that is p^B-optimal with $p^B \in K^B \backslash \mathbb{R}^n_+$, which ends the proof. $\square$

Proof of Proposition 3.2: (a) We have

$$\psi_P(x, y) = \begin{cases} \inf\{\lambda > 0 : \frac{1}{\lambda} y \in P(x)\} & \text{if } \frac{1}{\lambda} y \in P(x) \text{ for some } \lambda > 0 \\ \infty & \text{otherwise} \end{cases}$$

But from Proposition 2.3:

$$P(x) = \left\{ y \in \mathbb{R}^n_+ : p.y \le R(p,x), p \in \bigcup_{J \in \{\emptyset, B\}} K^J \right\}$$
$$= \bigcap_{p \in \bigcup_{J \in \{\emptyset, B\}} K^J} \left\{ y \in \mathbb{R}^n_+ : p.y \le R(p,x) \right\}.$$

Let us denote:

$$\psi_{OE}(x,y) = \inf\{\lambda > 0 : \frac{py}{\lambda} \le R(p,x)\}$$

Now, we have the equality:

$$\psi_P(x,y) = \inf\{\lambda > 0 : \frac{1}{\lambda} y \in P(x)\}$$
$$= \inf_{p \in \bigcup_{J \in \{\emptyset, B\}} K^J} \left\{ \inf\{\lambda > 0 : \frac{py}{\lambda} \le R(p,x)\} \right\}$$
$$= \inf_{p \in \bigcup_{J \in \{\emptyset, B\}} K^J} \psi_{OE}(x,y)$$

If $R(p,x) \neq 0$, an elementary calculus yields:

$$\psi_P(x,y) = \frac{p^* y}{R(p,x)}$$

Consequently,

$$\psi_P(x,y) = \inf_{p \in \bigcup_{J \in \{\emptyset, B\}} K^J} \psi_{OE}(x,y)$$
$$= \inf_{p \in \bigcup_{J \in \{\emptyset, B\}} K^J} \left\{ \frac{py}{R(p,x)} : R(p,x) \neq 0 \right\}.$$

(b) can be obtained in a way similar to Färe and Primont (1995). □

Proof of Proposition 3.3 : (a) follows from Proposition 2.5 and 3.2.b. (b) $P(x)$ satisfies cost disposability in the dimension of undesirable outputs if and only if $P^{\emptyset}(x) \neq P^{J}(x)$. However, for any output correspondence P, $y \in P(x)$ if and only if $\psi_P(x,y) \leq 1$. Moreover, by definition $P^J(x) \subset P^{\emptyset}(x)$ which implies that $\psi_{P^{\emptyset}}(x,y) \leq \psi_{P^J}(x,y)$. Hence, $P^{\emptyset}(x) \neq P^J(x)$ is equivalent to $\psi_{P^{\emptyset}}(x,y) \neq \psi_{P^J}(x,y)$. Consequently $P(x)$ satisfies cost disposability with the respect to residual outputs if and only if $\psi_{P^{\emptyset}}(x,y) < \psi_{P^J}(x,y)$ for at least some $y \in P(x)$, which ends the proof. $\Box$

Proof of Proposition 3.6: $(a) \Longrightarrow (b)$ is established in Varian (1984). Let us prove that $(b) \Longrightarrow (c)$. Define $P(x)$ as the smallest convex subset satisfying the B-disposal assumption and containing all the y^i such that $x^i \leq x$. Namely, if $A(x) = \{y^i : x^i \leq x, i \in I\}$, then we have:

$$P(x) = \Big(Co\big(A(x)\big) - K^{\emptyset}\Big) \cap \Big(Co\big(A(x)\big) - K^{B}\Big) \cap \mathbb{R}^n_+.$$

If there are no $x^i \leq x$, then let $P(x) = \emptyset$. Since $Co\big(A(x)\big)$ is a convex polytope, $\Big(Co\big(A(x)\big) - K^{I}\Big) \cap \mathbb{R}^n_+$ is a convex polytope for all $I \in \{\emptyset, B\}$. Therefore $P(x)$ is a convex polytope. Consequently, from Proposition 2.2, $P(x)$ satisfies an B-disposal assumption. Let us prove that $P(x)$ rationalizes the data. Since $P(x^i)$ is a convex polytope we only need to demonstrate this for the vertices of $P(x^i)$. But, the vertices of $P(x^i)$ are some subset of $A(x^i)$. Since all the x^k's in $A(x^i)$ satisfy the relevant condition by condition (b) we deduce that $P(x)$ rationalizes the data. Finally, by construction $P(x)$ is closed, convex and nested. The last implication $(c) \Longrightarrow (a)$ is obvious since (c) is stronger than (a). $\Box$

Proof of Proposition 4.1: (a) The subset $P^{J,DEA}$ is the intersection of a finite number of convex sets, thus it is convex. (b) holds by definition of $P^{J,DEA}$. (c) $P^{J,DEA}$ is intersection of a finite number of closed sets, hence it is closed. $\square$

Proof of Proposition 4.3: Based on Proof of Proposition 2.2 the result is immediate. $\square$