Exponential Environmental Productivity Index and Indicators

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Abstract

The major contributions of this paper are twofold. First, it introduces exponential environmental Luenberger productivity indicator and Malmquist-Luenberger productivity index constructed through an exponential distance function. Thereafter, an exponential version of the environmental Luenberger-Hicks-Moorsteen productivity indicator is proposed. Such a specification allows to overcome the special issue of infeasabilities. Second, looking from a dynamical viewpoint, we propose an exponential generalized dynamical distance function. This new efficiency measure shows the degree of efficiency of an observation, taking into consideration its technical efficiency and/or technological variation adjustment path. A sample of 11 representative French airports is considered over the period 2008-2011, in order to implement these new exponential environmental productivity index and indicators.

Keywords: Exponential Environmental Productivity Indicator, Exponential Environmental Productivity Index, Exponential Distance Function, Duality Theory, Generalized Dynamical Distance Function, Dynamical Deviation.

JEL: C61, D24, Q50

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1 Introduction

Two approaches of productivity measurement are often identified in the literature. On the one hand, following the initial work of Solow (1957), productivity growth in continuous time can be detected based on derivative of production function. On the other hand, productivity indices or indicators[1] allow to define productivity in consecutive time periods. Caves et al. (1982) introduced a theoretical analysis of the discrete time Malmquist (1953) input, output and productivity indices using distance functions as general representations of technology. Caves, Christensen and Diewert indices have been implemented in Pittman (1983) and Nishimizu and Page (1982), which first proposed to decompose productivity change into technical change and technical efficiency variation. Using Shephard (1970) distance function (i.e. the inverse of Farrell (1957) measure of technical efficiency), Färe et al. (1995) proposed to compute a Malmquist index, integrating the two-part Nishimizu and Page (1982) decomposition, based on nonparametric linear programming method.

Thereafter, Bjurek (1996) defines an alternative Hicks-Moorsteen index. This index corresponds to the ratio of a Malmquist output quantity index over a Malmquist input quantity index. Turning to the difference-based productivity measure, Chambers (2002) introduces a Luenberger productivity indicator. The latter is constructed as a difference-based index of directional distance functions, a generalisation of existing distance functions that accounts for both input contractions and output improvements (Briec, 1997). In the same vein, Briec and Kerstens (2004)[2] proposed a new difference-based variation on the Hicks-Moorsteen productivity index. The proposed Luenberger-Hicks-Moorsteen indicator is defined as a difference of Luenberger output quantity and Luenberger input quantity indicators, a generalisation of the Malmquist output and input quantity indices (Chambers et al., 1994).

[1]Throughout this paper, "indicators" are mentioned to define productivity measures based on differences and "indices" refer to ratio-based productivity measures. Note that, comparisons between ratio and difference approaches to index number theory from both a test and an economic perspective can be found in Chambers (1998, 2002) and Diewert (1998), among others.

[2]Chambers (1998) defines also an innovative productivity indicator using a special case of the shortage or directional distance function known as the translation function.

The above methods allow to define traditional productivity analysis (See for instance Boussemart et al., 2003; Färe et al., 1994; Kerstens and Van de Woestyne, 2014). Such methods do not take into consideration the impacts of environmental regulation on productivity change. More precisely, they appear limited when dealing with desirable and undesirable outputs to evaluate environmental impacts on technical change and technical efficiency variation. One alternative to the classical productivity analysis consists to use modified environmental productivity indicators and indices. Since the initial work of Pittman (1983), a growing literature about environmental productivity indicators and indices has been proposed (See for instance Tyteca, 1996; Aiken and Pasurka, 2003).

In line with the difference-based productivity indicators, Azad and Ancev (2014) and Picazo-Tadeo et al. (2014) proposed an environmental Luenberger productivity indicator. This method, based on the conceptual framework of the Luenberger productivity indicator, considers an output separation (desirable and undesirable) to assess productivity growth. Azad and Ancev (2014) considered an output (desirable and undesirable) oriented directional distance function (DDF) approach, while Picazo-Tadeo et al. (2014) used a subvector undesirable output oriented DDF. In the spirit of the ratio-based productivity index, Chung et al. (1997) introduced a Malmquist-Luenberger index. These authors used an adjusted environmental DDF within the traditional output oriented Malmquist index in order to show combined productive and environmental productivity changes. Dong-huyn Oh (2010) introduced a global Malmquist-Luenberger productivity index in order to overcome the special issue of infeasibilities. In tradition of the Hicks-Moorsteen productivity index, Färe et al. (2004) considered an environmental performance index defined as a ratio of Malmquist good output quantity index over Malmquist bad output quantity index. Recently, Abad (2015) proposed an innovative environmental generalised Luenberger-Hicks-Moorsteen productivity indicator and a new environmental generalised Hicks-Moorsteen productivity index allowing to overcome the issue of infeasabilities occuring when computing Malmquist-Luenberger and Luenberger productivity index and indicator.

In the above methods, environmental performance is evaluated using Data Envelopment Analysis (DEA) nonparametric linear programming models. Traditional environmental performance DEA models have mostly used static frameworks. Such approach allows to define combined productive and environmental performance for one period. Remark that some studies in nonparametric production analysis consider dynamic aspects (Fallah-Fini et al., 2014). In this paper, we propose to use both static and dynamical approaches. To this end, we consider a static and a dynamic exponential environmental distance function. We construct innovative exponential environmental productivity index and indicators based on the exponential distance function (Briec and Ravelojaona, 2015) allowing reduction of the input set and expansion of the output set simultaneously under a multiplicative technology which can take into account increasing marginal product. Looking from a dynamical viewpoint, we propose an exponential generalized dynamical distance function. This new efficiency measure shows the degree of efficiency of an observation, taking into consideration its technical efficiency and/or technological variation adjustment path. In addition, an "implicit" Törnqvist Total Factor Productivity is proposed through the dynamical exponential distance function following Törnqvist (1936). This index is supposed to be implicit due to the dissimilarity between the market prices weights and the weights proposed.

This note unfolds as follows. Technology and exponential distance function are defined in the next section. Section 3 introduces exponential environmental productivity index and indicators based on primal and dual viewpoint. Section 4 presents the exponential environmental distance function in both consecutive two time periods and dynamic frameworks. Moreover, dynamical deviation, exponential environmental productivity index and indicators in dynamical context and an implicit Törnqvist Total Factor Productivity is defined in this section. A sample of 11 representative French airports is considered over the period 2008-2011, in order to implement these new exponential environmental productivity index and indicators in section 5. Finally, Section 6 concludes, discusses limitations and, offers directions for future research.

2 Technology and Exponential Distance Function

2.1 Technology: Definition and Properties

The production technology transforms input vectors $x^t \in \mathbb{R}^n_+$ into output vectors $y^t = (y^t_d, y^t_b) \in \mathbb{R}^m_+$, with $m = m_d + m_b$, at the period t. Where, y^t_d is the desirable output and y^t_b is the undesirable output in environmental joint-production. The input correspondence, $L : \mathbb{R}^{m_d+m_b}_+ \longrightarrow 2^{\mathbb{R}^n_+}$, is defined as:

$$L(y^t) = \left\{x^t \in \mathbb{R}^n_+ : y^t \text{ can be produced by } x^t\right\}.$$

In this paper, we focus on the output correspondence, $P : \mathbb{R}^n_+ \longrightarrow 2^{\mathbb{R}^{m_d+m_b}_+}$, defined as follows:

$$P(x^t) = \left\{y^t \in \mathbb{R}^{m_d+m_b}_+ : x^t \text{ can produce } y^t\right\}.$$

L and P characterize the production technology and the graph technology T^t can be described by:

$$T^t = \left\{(x^t, y^t) \in \mathbb{R}^{n+m_d+m_b}_+ : x^t \in L(y^t) \vee y^t \in P(x^t)\right\}.$$

We assume that the production set, $P(x^t)$, satisfies the following regularity properties (see Hackman, 2008; Jacobsen, 1970; McFadden, 1978):

$P1$: $P(0) = \{0\}$ and $0 \in P(x^t)$ for all $x^t \in \mathbb{R}^n_+$.

$P2$: $P(x^t)$ is bounded above for all $x^t \in \mathbb{R}^n_+$.

$P3$: $P(x^t)$ is closed for all $x^t \in \mathbb{R}^n_+$.

$P4$: If $u \geq x \Rightarrow P(x) \supseteq P(u)$.

Remark that $P1$ postulates that there is no free lunch and that the null output can always be produced. Furthermore, $P2$ and $P3$ consisder that $P(x)$ is compact. Axiom $P4$ imposes the more traditional assumption of strong (or free) disposal of inputs.

Abad and Briec (2016) introduce an innovative B-disposal assumption, that is a kind of limited strong disposability. Based on this disposability assumption, these authors aim to model any Pollution-generating Technologies (PgT) in production processes compatible with a minimal set of assumptions. Authors provide an innovative axiomatic characterization of the incorrect modeling of variable returns-to-scale (VRS) assumption in traditional Shepard's weakly disposable technology (Kuosmanen, 2005; Leleu, 2013). Furthermore, considering a generalization of the B-disposability assumption on the technology, authors retrieve the by-production technology proposed by Murty et al. (2012). Let us postulate that the outputs satisfy the B-disposal assumption. We can define this assumption as follows:

Definition 2.1.1 *Let P be an output correspondence satisfying P1-P4. For all $y^t \in \mathbb{R}_+^m$, the output set $P(x^t)$ satisfies the B-disposal assumption if for all sets of output vectors $\left\{y^{tJ}\right\}_{J\in\{\emptyset,B\}} \subset P(x^t)$, $y^t \leq^J y^{tJ}$ for any $J \in \{\emptyset, B\}$ implies that $y^t \in P(x^t)$.*

Let $B \subset [m]$, indexing the bad outputs of the technology. We consider the following symbol:

$$y \geq^B v \iff \begin{cases} y_j \leq v_j & \text{if } j \in B \\ y_j \geq v_j & \text{else} \end{cases} \tag{2.1}$$

Remark that if $B = \emptyset$, then we retrieve the standard vector inequality. In this case, the B-disposal assumption reduces to the standard free disposability assumption.

2.2 Multiplicative Technology and Distance Function

2.2.1 Multiplicative Technology

The Data Envelopment Analysis (DEA) method estimation use the piecewise linear technology to set the efficiency frontier. However, this classic technology does not allow for increasing marginal products. Hence, following Banker and Maindiratta (1986) and Briec and Ravelojaona (2015), we introduce the multiplicative technology which is piecewise log-linear through a logarithmic transformation and takes into account increasing marginal product. This technology satisfies a "geometric convexity" such that, for all $(w_1^t, w_2^t) \in T^t$ and all $\mu_1, \mu_2 \geq 0$ with $\mu_1 + \mu_2 = 1$, $(w_1^t)^{\mu_1} \odot (w_2^t)^{\mu_2} \in T^t$.

Definition 2.2.1 *Consider J firms at the time period t,*

$$T^t = \{(x^t, y^t) \in \mathbb{R}_+^{n+m} : x^t \geq \prod_{k \in J} (x_k^t)^{\theta_k}, y^t \leq \prod_{k \in J} (y_k^t)^{\theta_k}, x^t \geq \prod_{k \in J} (x_k^t)^{\nu_k},$$
$$y^t \leq^B \prod_{k \in J} (y_k^t)^{\nu_k}, \nu \geq 0, \theta \geq 0\}$$

is called the multiplicative B-disposable non-parametric technology.

The multiplicative technology is non-linear such that to ease the estimation by a linear programming, a logarithmic transformation is applied.

Definition 2.2.2 *Consider J firms at the time period t. For $T_{++}^t = T^t \cap \mathbb{R}_{++}^{n+m}$:*

$$T_{\ln}^t = \{(\ln(x^t), \ln(y^t)) : (x^t, y^t) \in T_{++}^t\}$$
$$T_{\ln}^t = \{(x^t, y^t) \in \mathbb{R}_{++}^{n+m} : \ln(x^t) \geq \sum_{k \in J} \theta_k \ln(x_k^t), \ln(y^t) \leq \sum_{k \in J} \theta_k \ln(y_k^t),$$
$$\ln(x^t) \geq \sum_{k \in J} \nu_k \ln(x_k^t), \ln(y^t) \leq^B \sum_{k \in J} \nu_k \ln(y_k^t), \nu \geq 0, \theta \geq 0\}$$

is called the Neperian technology.

2.2.2 Exponential Distance Function

The deviation of the actual production set from the efficient production set means efficiency growth possibilities. Distance functions can measure this deviation length with respect to the technology. Instead of the radial measure of technical efficiency proposed in Banker and Maindiratta (1986), the exponential distance function introduced by Briec and Ravelojaona (2015) is applied. This distance function allows to reduce the input set and to expand the output set simultaneously[3] in a pre-assigned direction. This definition is closely related to the Directional distance function first proposed by Luenberger (1992) and by Chambers et al.(1996). Indeed, under a neperian technology, we retrieve the directional distance function allowing increasing marginal product. In addition, the exponential distance function is invariant with respect to the units measurement. Furthermore, it is connected to the Generalized multiplicative distance function presented by Mehdiloozad et al.(2014). For all $\delta^t \in \mathbb{R}$ and $(\alpha^t, \beta^t) \in [0,1]^n \times [0,1]^m$ the exponential distance function, $D^t_{\exp} : \mathbb{R}^{n+m}_+ \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$, is defined as follows:

$$D^t_{\exp}(x^t, y^t; \alpha^t, \beta^t) = \sup \left\{ \delta^t : \Phi^{\delta^t}_{\alpha^t,\beta^t}(x^t, y^t) \in T^t \right\}.$$

Where the linear map $\Phi^{\delta^t}_{\alpha^t,\beta^t} : \mathbb{R}^{n+m}_+ \longrightarrow \mathbb{R}^{n+m}_+$ is defined for some $\delta^t \in \mathbb{R}$ as follows:

$$\Phi^{\delta^t}_{\alpha^t,\beta^t}(x^t, y^t) = (e^{-\delta^t A^t} x^t, e^{\delta^t B^t} y^t).$$

In the above definition of $\Phi^{\delta^t}_{\alpha^t,\beta^t}(x^t, y^t)$, A^t and B^t are respectively two diagonal matrices such that $A^t = diag(\alpha^t)$ and $B^t = diag(\beta^t)$.

Following Briec and Ravelojaona (2015), we assume that this distance function satisfies the properties quoted below:

D1: $D^t_{\exp}(x^t, y^t; \alpha^t, \beta^t) \geq 0 \Leftrightarrow (x^t, y^t) \in T^t$.

[3] Remark that we can consider input or output oriented exponential distance function.

D2: For $(u^t, v^t), (x^t, y^t) \in T^t$, if $(-u^t, v^t) \geq (-x^t, y^t)$ then $D^t_{\exp}(x^t, y^t; \alpha^t, \beta^t) \leq D^t_{\exp}(u^t, v^t; \alpha^t, \beta^t)$.

D3: $D^t_{\exp}(\Phi^{\theta^t}_{\alpha^t,\beta^t}(x^t, y^t); \alpha^t, \beta^t) = D^t_{\exp}(x^t, y^t; \alpha^t, \beta^t) - \theta^t$.

D4: $D^t_{\exp}(x^t, y^t; \alpha^t, \beta^t) = 0 \Leftrightarrow (x^t, y^t) \in \partial_{\alpha^t,\beta^t}(T^t)$, the efficient technology subset.

D5: $D^t_{\exp}(x^t, y^t; \alpha^t, \beta^t)$ is commensurable.

D1 asserts that the exponential distance function represents the technology. D2 postulates the monotonicity of the function. D3 states that the exponential distance function is translation homothetic. D4 identifies the efficient technology subset and D5 claims that the function satisfies the commensurability condition stated by Russell (1988).

Through a logarithmic transformation, this exponential distance function is analogous to the directional distance function under a neperian technology $T^t_{\ln}$, such that :

$$\ln(\Phi^{\delta^t}_{\alpha^t,\beta^t}(x^t, y^t)) = \ln(e^{-\delta^t\alpha^t}x^t, e^{\delta^t\beta^t}y^t) = (\ln(x^t) - \delta^t\alpha^t, \ln(y^t) + \delta^t\beta^t).$$

Let the exponential distance function, $D^t_{\exp} : \mathbb{R}^{n+m}_+ \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$, be the map defined by:

$$\begin{aligned} D^t_{\exp}(x^t, y^t; \alpha^t, \beta^t) &= \sup\left\{\delta^t : \ln(\Phi^{\delta^t}_{\alpha^t,\beta^t}(x^t, y^t)) \in T^t_{\ln}\right\} \\ &= \sup\left\{\delta^t : (\ln(x^t) - \delta^t\alpha^t, \ln(y^t) + \delta^t\beta^t) \in T^t_{\ln}\right\} \\ &= \sup\left\{\delta^t : (u^t - \delta^t\alpha^t, v^t + \delta^t\beta^t) \in T^t_{\ln}\right\} \\ &= D^t_{\ln}(u^t, v^t; \alpha^t, \beta^t). \end{aligned}$$

This neperian distance function is assumed to satisfy the usual axioms of directional distance function defined by Chambers et al. (1998).

Following Chung et al. (1997), we introduce the environmental exponential distance function which can be applied to environmental issues such that both good and bad outputs are simultaneously increased and reduced. To do so, we present the environmental exponential distance function output-oriented (ED) with $\alpha = 0$ and $\beta = (\beta_d^t, \beta_b^t) \in [0,1]$. This function, $ED_{\exp^O} : \mathbb{R}_+^{m_d+m_b} \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$, is defined as below:

$$ED^t_{\exp^O}(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t) = \sup\left\{\delta^t : (e^{-\delta^t\beta_b^t} y_b^t,\ e^{\delta^t\beta_d^t} y_d^t) \in P(x^t)\right\}. \tag{2.2}$$

As shown above, the ED can be expressed as follows:

$$\begin{aligned} ED^t_{\exp^O}(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t) &= \sup\left\{\delta^t : \ln(e^{-\delta^t\beta_b^t} y_b^t,\ e^{\delta^t\beta_d^t} y_d^t) \in P^t_{\ln}\right\} \\ &= \sup\left\{\delta^t : (\ln(y_b^t) - \delta^t\beta_b^t,\ \ln(y_d^t) + \delta^t\beta_d^t) \in P^t_{\ln}\right\} \\ &= \sup\left\{\delta^t : (v_b^t - \delta^t\beta_b^t,\ v_d^t + \delta^t\beta_d^t) \in P^t_{\ln}\right\} \\ &= ED^t_{\ln^O}(u^t, v_b^t, v_d^t; 0, \beta_b^t, \beta_d^t). \end{aligned}$$

In the above equations $P^t_{\ln} = \left\{(\ln(y_b^t), \ln(y_d^t)) : (y_b^t, y_d^t) \in P^t_{++}\right\}$, with $P^t_{++} = P^t \cap \mathbb{R}_{++}^{m_b+m_d}$.

3 Exponential Environmental Index and Indicators

3.1 Exponential Malmquist-Luenberger Productivity Index

Productivity indices permit to estimate the performance evolution of a firm between two periods. Several environmental productivity indices were introduced these last decades (Chung et al., 1997; Färe et al., 2004). Using the output-oriented Malmquist productivity index, Chung et al. (1997) developed the Malmquist-Luenberger (ML) productivity index. These authors use the fact that the output distance function is a special case of the output-oriented directional distance function (Chambers et al., 1996). From this result, they introduce the latter within the standard output-oriented Malmquist productivity index [4]. The Exponential

[4] In the tradition of the Hicks-Moorsteen productivity index, Färe et al. (2004) introduce an Environmental Performance Index (EPI) defined as the ratio of a Malmquist good output quantity index (MG) over a

Malmquist-Luenberger (EML) productivity index is defined as follows:

$$\begin{aligned} &EML(x^t, x^{t+1}, y_d^t, y_b^t, y_d^{t+1}, y_b^{t+1}; 0, \gamma^t, \gamma^{t+1}) \\ &= \left[\frac{1+ED^t_{\exp O}(x^t, y_d^t, y_b^t; \gamma^t)}{1+ED^t_{\exp O}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; \gamma^{t+1})} \times \frac{1+ED^{t+1}_{\exp O}(x^t, y_d^t, y_b^t; \gamma^t)}{1+ED^{t+1}_{\exp O}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; \gamma^{t+1})} \right]^{1/2}. \end{aligned} \tag{3.1}$$

Through a logarithmic transformation of the ED defined above, we have:

$$\begin{aligned} &ML_{\ln}(u^t, u^{t+1}, v_d^t, v_b^t, v_d^{t+1}, v_b^{t+1}; \gamma^t, \gamma^{t+1}) \\ &= \left[\frac{1+ED^t_{\ln O}(u^t, v_d^t, v_b^t; \gamma^t)}{1+ED^t_{\ln O}(u^{t+1}, v_d^{t+1}, v_b^{t+1}; \gamma^{t+1})} \times \frac{1+ED^{t+1}_{\ln O}(u^t, v_d^t, v_b^t; \gamma^t)}{1+ED^{t+1}_{\ln O}(u^{t+1}, v_d^{t+1}, v_b^{t+1}; \gamma^{t+1})} \right]^{1/2}, \end{aligned} \tag{3.2}$$

where $\gamma^t = (0, \beta_d^t, \beta_b^t)$ and such that:

$$ED^t_{\exp O}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; \gamma^{t+1}) = \sup \left\{ \delta^{t+1} : (e^{-\delta^{t+1}\beta_b^{t+1}} y_b^{t+1},\ e^{\delta^{t+1}\beta_d^{t+1}} y_d^{t+1}) \in P(x^t) \right\}.$$

Remark that when $\gamma^t = \left(0, \ln(y_d^t), -\ln(y_b^t)\right)$ the EML corresponds to the exponential version of the Malmquist output index. As in the case of the Malmquist output index, ecological productivity improvement, respectively deterioration, takes place when values of the EML productivity index is above, respectively below, unity. Furthermore, the EML can be decomposed in two components: efficiency variation (EMLEV) and technical change (EMLTC).

$$EMLEV(x^t, x^{t+1}, y_d^t, y_b^t, y_d^{t+1}, y_b^{t+1}; 0, \gamma_t, \gamma t+1) = \frac{1+ED^t_{\exp O}(x^t, y_d^t, y_b^t; \gamma_t)}{1+ED^{t+1}_{\exp O}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; \gamma_{t+1})} \tag{3.3}$$

and

Malmquist bad output quantity index (MB) at base period t.

$$EMLTC(x^t, x^{t+1}, y_d^t, y_b^t, y_d^{t+1}, y_b^{t+1}; 0, \gamma_t, \gamma t+1) = \left[\frac{1 + ED^{t+1}_{\exp^O}(x^t, y_d^t, y_b^t; \gamma_t)}{1 + ED^{t}_{\exp^O}(x^t, y_d^t, y_b^t; \gamma_t)} \times \frac{1 + ED^{t+1}_{\exp^O}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; \gamma_{t+1})}{1 + ED^{t}_{\exp^O}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; \gamma_{t+1})} \right]^{1/2}. \tag{3.4}$$

3.2 Exponential Environmental Luenberger Productivity Indicator

Let us consider the Luenberger productivity indicator applied to the environmental area. Following Azad and Ancev (2014) and Abad (2015), we propose to use the Environmental Luenberger productivity indicator based upon the ED. Thus, the exponential environmental Luenberger (EEL) productivity indicator is described as following:

$$\begin{aligned} &EEL(x^t, x^{t+1}, y_d^t, y_b^t, y_d^{t+1}, y_b^{t+1}; \gamma^t, \gamma^{t+1}) \\ &= \frac{1}{2}\Big[\big(ED^t_{\exp^O}(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t) - ED^t_{\exp^O}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; 0, \beta_d^{t+1}, \beta_b^{t+1})\big) \\ &\quad + \big(ED^{t+1}_{\exp^O}(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t) - ED^{t+1}_{\exp^O}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; 0, \beta_d^{t+1}, \beta_b^{t+1})\big)\Big]. \end{aligned} \tag{3.5}$$

Through a logarithmic transformation of the ED defined above, we have:

$$\begin{aligned} &EL_{\ln}(u^t, u^{t+1}, v_d^t, v_b^t, y_d^{t+1}, y_b^{t+1}; \gamma^t, \gamma^{t+1}) \\ &= \frac{1}{2}\Big[\big(ED^t_{\ln^O}(u^t, v_d^t, v_b^t; 0, \beta_d^t, \beta_b^t) - ED^t_{\ln^O}(u^{t+1}, v_d^{t+1}, v_b^{t+1}; 0, \beta_d^{t+1}, \beta_b^{t+1})\big) \\ &\quad + \big(ED^{t+1}_{\ln^O}(u^t, v_d^t, v_b^t; 0, \beta_d^t, \beta_b^t) - ED^{t+1}_{\ln^O}(u^{t+1}, v_d^{t+1}, v_b^{t+1}; 0, \beta_d^{t+1}, \beta_b^{t+1})\big)\Big], \end{aligned} \tag{3.6}$$

where $\gamma^t = (0, \beta_d^t, \beta_b^t)$ and such that:

$$ED^t_{\exp^O}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; \gamma^{t+1}) = \sup\left\{\delta^{t+1} : (e^{-\delta^{t+1}\beta_b^{t+1}} y_b^{t+1},\ e^{\delta^{t+1}\beta_d^{t+1}} y_d^{t+1}) \in P(x^t)\right\}.$$

The exponential environmental Luenberger productivity indicator can be decomposed as follows (Azad and Ancev, 2014):

$$\begin{aligned}
&EEL(x^t, x^{t+1}, y_d^t, y_b^t, y_d^{t+1}, y_b^{t+1}; \gamma^t, \gamma^{t+1}) \qquad (3.7)\\
&= \Big[ED^t_{\exp^O}(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t) - ED^{t+1}_{\exp^O}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; 0, \beta_d^{t+1}, \beta_b^{t+1})\Big]\\
&\quad + \frac{1}{2}\Big[\big(ED^{t+1}_{\exp^O}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; 0, \beta_d^{t+1}, \beta_b^{t+1}) - ED^t_{\exp^O}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; 0, \beta_d^{t+1}, \beta_b^{t+1})\big)\\
&\quad + \big(ED^{t+1}_{\exp^O}(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t) - ED^t_{\exp^O}(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t)\big)\Big].
\end{aligned}$$

Ratio in the first bracket depicts efficiency variation among base years t and $t+1$. Term in the second bracket shows technological change among base years t and $t+1$.

3.3 Exponential Environmental Luenberger-Hicks-Moorsteen Productivity Indicator

Abad (2015) introduces both environmental generalised Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity index and indicator. These innovative ratio-based and difference-based productivity measures focus on desirable and undesirable quality attributes. Let us consider an exponential version of the environmental generalised Luenberger-Hicks-Moorsteen productivity indicator. Most precisely, we consider an Exponential version of the subvector Undesirable Component Environmental Generalised productivity indicator (EUCEG). Such a representation solely focus on desirable and undesirable outputs, allowing to overcome the special issue of infeasabilities (Briec and Kerstens, 2009) in the computation of Malmquist-Luenberger productivity index and Environmental Luenberger productivity indicator (Figure

1.).

In the same vein of Luenberger-Hicks-Moorsteen indicator, the EUCEG is defined as the difference between an Exponential Environmental Luenberger Output ($EELO^t$) quantity indicator and a subvector undesirable component Exponential Environmental Luenberger Input ($EELI^t$) quantity indicator:

$$\begin{aligned} & EUCEG^t(x^t, y_d^t, y_b^t, y_d^{t+1}, y_b^{t+1}; \xi^t, \psi^t, \xi^{t+1}\psi^{t+1}) \\ &= EELO^t(x^t, y_d^t, y_b^t, y_d^{t+1}; \xi^t, \xi^{t+1}) - EELI^t(x^t, y_d^t, y_b^t, y_b^{t+1}; \psi^t, \psi^{t+1}), \end{aligned} \tag{3.8}$$

where:

$$\begin{aligned} & EELO^t(x^t, y_d^t, y_b^t, y_d^{t+1}; \xi^t, \xi^{t+1}) \\ &= ED^t_{\exp O}(x^t, y_d^t, y_b^t; 0, \beta_d^t, 0) - ED^t_{\exp O}(x^t, y_d^{t+1}, y_b^t; 0, \beta_d^{t+1}, 0) \end{aligned}$$

and

$$\begin{aligned} & EELI^t(x^t, y_d^t, y_b^t, y_b^{t+1}; \psi^t, \psi^{t+1}) \\ &= ED^t_{\exp O}(x^t, y_d^t, y_b^{t+1}; 0, 0, \beta_b^{t+1}) - ED^t_{\exp O}(x^t, y_d^t, y_b^t; 0, 0, \beta_b^t). \end{aligned}$$

Also a base period $t+1$ subvector Exponential Undesirable Component Environmental Generalised productivity indicator ($EUCEG^{t+1}$) can be similarly defined as follows:

$$\begin{aligned} & EUCEG^{t+1}(x^t, y_d^t, y_b^t, y_d^{t+1}, y_b^{t+1}; \xi^t, \xi^{t+1}, \psi^t, \psi^{t+1}) \\ &= EELO^{t+1}(x^{t+1}, y_d^t, y_b^{t+1}, y_d^{t+1}; \xi^t, \xi^{t+1}) - EELI^{t+1}(x^{t+1}, y_d^{t+1}, y_b^t, y_b^{t+1}; \psi^t, \psi^{t+1}), \end{aligned} \tag{3.9}$$

where:

$$\begin{aligned} & EELO^{t+1}(x^{t+1}, y_d^t, y_b^{t+1}, y_d^{t+1}; \xi^t, \xi^{t+1}) \\ &= ED^{t+1}_{\exp O}(x^{t+1}, y_d^t, y_b^{t+1}; 0, \beta_d^t, 0) - ED^{t+1}_{\exp O}(x^{t+1}, y_b^{t+1}, y_d^{t+1}; 0, \beta_d^{t+1}, 0) \end{aligned}$$

and:

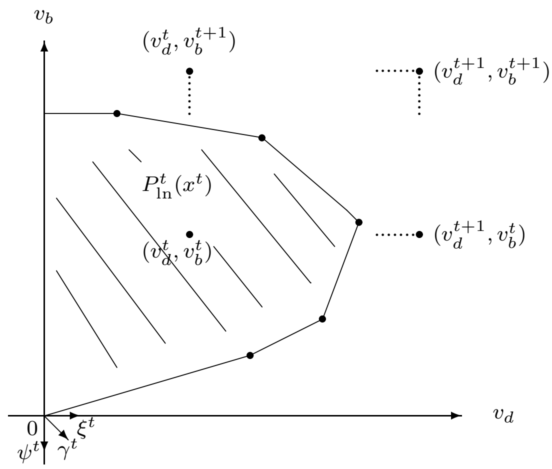

Figure 1: EUCEG and the issue of infeasibilities

$$
\begin{aligned}
&EELI^{t+1}(x^{t+1}, y_d^{t+1}, y_b^t, y_b^{t+1}; \psi^t, \psi^{t+1}) \\
&= ED^{t+1}_{\exp O}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; 0, 0, \beta_b^{t+1}) - ED^{t+1}_{\exp O}(x^{t+1}, y_d^{t+1}, y_b^t; 0, 0, \beta_b^t).
\end{aligned}
$$

An arithmetic mean of these two base periods exponential environmental generalized Luenberger-Hicks-Moorsteen indicators is:

$$
\begin{aligned}
&EUCEG^{t,t+1}(x^t, y_d^t, y_b^t, y_d^{t+1}, y_b^{t+1}; \xi^t, \xi^{t+1}, \psi^t, \psi^{t+1}) \\
&= \tfrac{1}{2}[EUCEG^t(x^t, y_d^t, y_b^t, y_d^{t+1}, y_b^{t+1}; \xi^t, \xi^{t+1}, \psi^t, \psi^{t+1}) \\
&\quad + EUCEG^{t+1}(x^t, y_d^t, y_b^t, y_d^{t+1}, y_b^{t+1}; \xi^t, \xi^{t+1}, \psi^t, \psi^{t+1})].
\end{aligned} \tag{3.10}
$$

The EUCEG productivity indicator shows combined desirable and undesirable outputs productivity improvement, respectively decline, when it takes positive, respectively negative, values. The productivity change in desirable outputs is greater than the changes in productivity associated with the undesirable outputs. The innovative EUCEG productivity indicator takes the form of an additive complete Total Factor Productivity (TFP) measure. Following O'Donnell (2010, 2012, 2014, 2016), the EUCEG could be decomposed within finer measures of efficiency in terms of aggregate quantities.

3.4 Exponential Environmental Productivity Indicators and Duality Theory

To study these new multiplicative environmental productivity index and indicators from a dual standpoint, we introduce a Cobb-Douglas revenue function, $R_{CD} : \mathbb{R}^{n+m_d+m_b}_{++} \longrightarrow \mathbb{R} \cup \{-\infty\}$, and a Neperian revenue function, $R_{\ln} : \mathbb{R}^{n+m_d+m_b}_{++} \longrightarrow \mathbb{R} \cup \{-\infty\}$, defined by:

$$R_{CD}(p_d^t, p_b^t, x^t) = \begin{cases} \sup\limits_{y_d^t, y_b^t} \left\{ \prod_{i \in m_d} y_i^{t p_i^t} \prod_{j \in m_b} y_j^{t p_j^t} : (y_d^t, y_b^t) \in P(x^t) \right\} & \text{if } P(x^t) \neq \emptyset \\ -\infty & \text{if } P(x^t) = \emptyset \end{cases} \tag{3.11}$$

And

$$R_{\ln}(p_d^t, p_b^t, u^t) = \begin{cases} \sup\limits_{\upsilon_d^t, \upsilon_b^t} \left\{ p_d^t \upsilon_d^t + p_b^t \upsilon_b^t : \upsilon_d^t, \upsilon_b^t \in P_{\ln}^t \right\} & \text{if } P_{\ln}^t \neq \emptyset \\ -\infty & \text{if } P_{\ln}^t = \emptyset \end{cases} \tag{3.12}$$

Note that:

$$
\begin{aligned}
R_{CD}(p_d^t, p_b^t, x^t) &\equiv \exp\big(R_{\ln}(p_d^t, p_b^t, u^t)\big)\\
&\equiv \exp\Big(\sup_{\upsilon_d^t,\upsilon_b^t}\{p_d^t\upsilon_d^t + p_b^t\upsilon_b^t : \upsilon_d^t, \upsilon_b^t \in P_{\ln}^t\}\Big)\\
&\equiv \sup_{\upsilon_d^t,\upsilon_b^t}\big\{\exp(p_d^t\upsilon_d^t + p_b^t\upsilon_b^t) : \upsilon_d^t, \upsilon_b^t \in P_{\ln}^t\big\}\\
&\equiv \sup_{y_d^t,y_b^t}\Big\{\exp\big(p_d^t\ln(y_d^t) + p_b^t\ln(y_b^t)\big) : \upsilon_d^t, \upsilon_b^t \in P_{\ln}^t\Big\}\\
&\equiv \sup_{y_d^t,y_b^t}\Big\{\exp\big(\sum_{i\in m_d}\ln(y_i^{tp_i^t}) + \sum_{j\in m_b}\ln(y_j^{tp_j^t})\big) : \upsilon_d^t, \upsilon_b^t \in P_{\ln}^t\Big\}\\
&\equiv \sup_{y_d^t,y_b^t}\Big\{\exp\big(\ln(\prod_{i\in m_d}y_i^{tp_i^t}) + \ln(\prod_{j\in m_b}y_j^{tp_j^t})\big) : \upsilon_d^t, \upsilon_b^t \in P_{\ln}^t\Big\}\\
&\equiv \sup_{y_d^t,y_b^t}\Big\{\exp\big(\ln(\prod_{i\in m_d}y_i^{tp_i^t}\prod_{j\in m_b}y_j^{tp_j^t})\big) : \upsilon_d^t, \upsilon_b^t \in P_{\ln}^t\Big\}\\
&\equiv \sup_{y_d^t,y_b^t}\big\{\prod_{i\in m_d}y_i^{tp_i^t}\prod_{j\in m_b}y_j^{tp_j^t} : y_d^t, y_b^t \in P^t\big\}.
\end{aligned}
$$

In line with the initial work of Luenberger (1992), Chambers et al. (1996) and Briec and Ravelojaona (2015), one can establish the following duality result between the environmental exponential distance function and the revenue function:

$$
ED^t_{\exp^O}(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t) = \ln\Big(\inf_{p_d^t,p_b^t}\big\{\frac{R_{\exp}(p_d^t, p_b^t, x^t)}{(\prod_{i\in m_d}y_i^{tp_i^t}\prod_{j\in m_b}y_j^{tp_j^t})} : p_d^t\beta_d^t + p_b^t\beta_b^t = 1\big\}\Big) \tag{3.13}
$$

and

$$
R_{CD}(p_d^t, p_b^t, x^t) = \ln\Big(\sup_{y_d^t,y_b^t}\big\{\prod_{i\in m_d}y_i^{tp_i^t}\prod_{j\in m_b}y_j^{tp_j^t}ED^t_{\exp^O}(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t)^{p_d^t\beta_d^t + p_b^t\beta_b^t}\big\}\Big). \tag{3.14}
$$

It is consistent to point out that the revenue function (R_{CD}) has a Cobb-Douglas functional form. However, R_{CD} is not the revenue function related to a Cobb-Douglas production function but is the hyperplane support of the multiplicative technology set. The intersection of those hyperplanes constitute a boundary which is the subset of overall efficient units. The distance between this boundary and the production units means technical and allocative inefficiencies.

Following Luenberger (1996), the adjusted output price function is the point to set map $(p_d^t, p_b^t) : \mathbb{R}^n_{++} \times \mathbb{R}^{m_d+m_b}_{++} \times [0,1]^{m_d} \times [0,1]^{m_b} \longrightarrow 2^{\mathbb{R}^{m_d}_+ \times \mathbb{R}^{m_b}}$ defined by:

$$(p_d^t, p_b^t)(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t) = \arg \min_{(p_d^t, p_b^t) \in \mathbb{R}^{m_d}_+ \times \mathbb{R}^{m_b}} \Big\{ \frac{R_{\exp}(p_d^t, p_b^t, x^t)}{(\prod_{i \in m_d} y_i^{t p_i^t} \prod_{j \in m_b} y_j^{t p_j^t})} : p_d^t \beta_d^t + p_b^t \beta_b^t = 1 \Big\}. \tag{3.15}$$

At points where the exponential distance function is differentiable, an immediate consequence of envelope theorem gives:

$$p_d^t(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t) = -\frac{\partial ED^t_{\exp^O}(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t)}{\partial ln(y_d^t)}. \tag{3.16}$$

and

$$p_b^t(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t) = -\frac{\partial ED^t_{\exp^O}(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t)}{\partial ln(y_b^t)}. \tag{3.17}$$

Remark that the use of the B-disposal assumption to model pollution-generating technologies allows to consider negative shadow prices for the bad outputs (Abad and Briec, 2016).

4 Distance Function in Dynamical Context

4.1 Discrete Time with Two Periods

Suppose two time periods $\tau \in \{t, t-1\}$ and that the firm is inefficient at $\tau = t-1$ with production units $(u^{t-1}, v_b^{t-1}, v_d^{t-1})$, where $v_b = \{v_{b,1}, ..., v_{b,i}\}$ is the undesirable output vector and $v_d = \{v_{d,1}, ..., v_{d,k}\}$ is the desirable output vector such that $i \in [m_b]$ and $k \in [m_d]$. The neperian distance function at period $\tau = t-1$ with respect to the production technology at period t is defined as follows:

$$ED^{t(t-1)}_{\ln^O}(u^{t-1}, v_b^{t-1}, v_d^{t-1}; 0, \beta_b^{t-1}, \beta_d^{t-1}) = \sup\left\{\delta : (v_b^{t-1} - \delta\beta_b^{t-1}, v_d^{t-1} + \delta\beta_d^{t-1}) \in P^t_{ln}\right\}.$$

Proposition 4.1.1 *For all* $(u^t, v_b^t, v_d^t) \in T^{t(t-1)}_{\ln}$ *and for all* $(0, \beta_d^{t-1}, \beta_b^{t-1}) \in [0,1]$, *it can be established that:*

$$e^{(\delta^{t(t-1)})} = \left(\frac{y_d^t}{y_d^{t-1}}\right)^{\frac{1}{\rho_d^{t(t-1)}\beta_d^{t-1}}} = \left(\frac{y_b^{t-1}}{y_b^t}\right)^{\frac{1}{\rho_b^{t(t-1)}\beta_b^{t-1}}}. \tag{4.1}$$

Proposition 4.1.2 *For all* $(u^{t-1}, v_b^{t-1}, v_d^{t-1}) \in T^{t-1(t)}_{\ln}$ *and* $(0, \beta_b^t, \beta_d^t) \in [0,1]$, *the neperian and the exponential distance functions can be expressed as variations of the factors or of the products:*

$$e^{(\delta^{t-1(t)})} = \left(\frac{y_d^t}{y_d^{t-1}}\right)^{\frac{1}{\rho_d^{t-1(t)}\beta_d^t}} = \left(\frac{y_b^{t-1}}{y_b^t}\right)^{\frac{1}{\rho_b^{t-1(t)}\beta_b^t}}. \tag{4.2}$$

Propositions **??** and **??** are proved in Appendix A. The dynamical exponential distance function can be defined as a ratio of vectors through the introduction of the parameter ρ (see Subsection 4.3).

4.2 Generalized Dynamical Distance Function

The notion established above can be extend to several periods. Suppose that $\tau \in \{1, ..., t\}$ such that the production units (u^t, v_b^t, v_d^t) at $\tau = t$ were adjusted all along in the same proportion $\rho^t \in \mathbb{R}$ of each distance function, from $ED^{2(1)}_{\ln^O}(u^1, v_b^1, v_d^1; 0, \beta_b^1, \beta_d^1)$ to $ED^{t-1(t-2)}_{\ln^O}(u^{t-2}, v_b^{t-2}, v_d^{t-2}; 0, \beta_b^{t-2}, \beta_d^{t-2})$. Assume that $\beta_b^1 = \beta_b^2 = ... = \beta_b^t$ and $\beta_d^1 = \beta_d^2 = ... = \beta_d^t$. Suppose also that the distance function of the penultimate period $\tau = t-1$ is used to adjust the production units in $\rho^{t(t-1)}\delta^{t(t-1)}$ proportion.

Proposition 4.2.1 *For all* $(u^t, v_b^t, v_d^t) \in T_{\ln}^{t(t-1)}$ *and* $(0, \beta_b^{t-1}, \beta_d^{t-1}) \in [0,1]$*, the neperian and the exponential distance functions can be expressed as variations of the factors or of the products:*

$$e^{\delta^{t(t-1)}} = \left(\frac{y_b^1}{y_b^t}\right)^{\frac{1}{\rho_b^{t(t-1)}\beta_b^{t-1}}} \times \prod_{r=2}^{t-1} e^{-\frac{\rho_b^{r(r-1)}}{\rho_b^{t(t-1)}}\delta^{r(r-1)}} = \left(\frac{y_d^t}{y_d^1}\right)^{\frac{1}{\rho_d^{t(t-1)}\beta_d^{t-1}}} \times \prod_{r=2}^{t-1} e^{-\frac{\rho_d^{r(r-1)}}{\rho_d^{t(t-1)}}\delta^{r(r-1)}}. \quad (4.3)$$

Proposition 4.2.2 *For all* $(u^{t-1}, v_b^{t-1}, v_d^{t-1}) \in T_{\ln}^{t-1(t)}$ *and* $(0, \beta_b^t, \beta_d^t) \in [0,1]$*, the neperian and the exponential distance functions can be expressed as variations of the factors or of the products:*

$$e^{\delta^{t-1(t)}} = \left(\frac{y_b^1}{y_b^t}\right)^{\frac{1}{\rho_b^{t-1(t)}\beta_b^{t}}} \times \prod_{r=1}^{t-1} e^{-\frac{\rho_b^{r(r+1)}}{\rho_b^{t-1(t)}}\delta^{r(r+1)}} = \left(\frac{y_d^t}{y_d^1}\right)^{\frac{1}{\rho_d^{t-1(t)}\beta_d^{t}}} \times \prod_{r=1}^{t-1} e^{-\frac{\rho_d^{r(r+1)}}{\rho_d^{t-1(t)}}\delta^{r(r+1)}}. \quad (4.4)$$

Proofs of propositions **??** and **??** are in Appendix B.

4.3 Dynamical deviation

It is seen that the distance function in dynamical context contains a parameter depicted by ρ. This latter can represent both internal and external constraints which does not allow the complete achievement of efficiency variation and technological change by the firm. External constraints, such externalities, environmental policies, economic circumstances etc., cannot be controlled by the firms, while internal constraints depend essentially on their production processes (technological competitiveness, etc.) and managerial procedures. Therefore, firms can affect these variables adopting new investments, promoting innovative staff members skills etc.. All of the times the dynamical deviation ρ is greater, respectively lesser, than zero, both internal and external constraints not allow optimal technical efficiency adjustment and/or technological variation of the firms. Furthermore, when the dynamical deviation ρ is equal to one, both internal and external constraints not affect these adjustments of the firms. Since the constraints affecting the adjustements are not the same in the desirable and undesirable output dimensions, dynamical deviation ρ can be defined separately in both dimensions (Figure **??**).

Proposition 4.3.1 *For all* $i \in \{2, ..., t\}$ *and* $\delta^{t(t-1)}$*, the externality weight is expressed as follows:*

$$\rho_b^{i(i-1)} = \frac{v_b^1 - v_b^t}{\delta^{i(i-1)}\beta_b^{i-1}} - \sum_{r=2}^{i-1} \rho_b^{r(r-1)} \frac{\delta^{r(r-1)}}{\delta^{i(i-1)}} - \sum_{r=i+1}^{t} \rho_b^{r(r-1)} \frac{\delta^{r(r-1)}}{\delta^{i(i-1)}} \tag{4.5}$$

and

$$\rho_d^{i(i-1)} = \frac{v_d^t - v_d^1}{\delta^{i(i-1)}\beta_d^{i-1}} - \sum_{r=2}^{i-1} \rho_d^{r(r-1)} \frac{\delta^{r(r-1)}}{\delta^{i(i-1)}} - \sum_{r=i+1}^{t} \rho_d^{r(r-1)} \frac{\delta^{r(r-1)}}{\delta^{i(i-1)}}. \tag{4.6}$$

Thus, it is obvious that the exponential expression of the parameter is:

$$e^{\rho_b^{i(i-1)}} = \left(\frac{y_b^1}{y_b^t}\right)^{\frac{1}{\delta^{i(i-1)}\beta_b^{i-1}}} \times \Pi_{r=2}^{i-1} e^{-\rho_b^{r(r-1)} \frac{\delta^{r(r-1)}}{\delta^{i(i-1)}}} \times \Pi_{r=i+1}^{t} e^{-\rho_b^{r(r-1)} \frac{\delta^{r(r-1)}}{\delta^{i(i-1)}}}$$

and

$$e^{\rho_d^{i(i-1)}} = \left(\frac{y_d^t}{y_d^1}\right)^{\frac{1}{\delta^{i(i-1)}\beta_d^{i-1}}} \times \Pi_{r=2}^{i-1} e^{-\rho_d^{r(r-1)} \frac{\delta^{r(r-1)}}{\delta^{i(i-1)}}} \times \Pi_{r=i+1}^{t} e^{-\rho_d^{r(r-1)} \frac{\delta^{r(r-1)}}{\delta^{i(i-1)}}}.$$

Proposition 4.3.2 *For all* $i \in \{1, ..., t-1\}$ *and* $\delta^{t(t+1)}$*, the externality weight is expressed as follows:*

$$\rho_b^{i(i+1)} = \frac{v_b^1 - v_b^t}{\delta^{i(i+1)}\beta_b^{i+1}} - \sum_{r=1}^{i-1} \rho_b^{r(r+1)} \frac{\delta^{r(r+1)}}{\delta^{i(i+1)}} - \sum_{r=i+1}^{t-1} \rho_b^{r(r+1)} \frac{\delta^{r(r+1)}}{\delta^{i(i+1)}} \tag{4.7}$$

and

$$\rho_d^{i(i+1)} = \frac{v_d^t - v_d^1}{\delta^{i(i+1)}\beta_d^{i+1}} - \sum_{r=1}^{i-1} \rho_d^{r(r+1)} \frac{\delta^{r(r+1)}}{\delta^{i(i+1)}} - \sum_{r=i+1}^{t-1} \rho_d^{r(r+1)} \frac{\delta^{r(r+1)}}{\delta^{i(i+1)}}. \tag{4.8}$$

And through exponential transformation, we have:

$$e^{\rho_b^{i(i+1)}} = \left(\frac{y_b^1}{y_b^t}\right)^{\frac{1}{\delta^{i(i+1)}\beta_b^{i+1}}} \times \Pi_{r=1}^{i-1} e^{-\rho_b^{r(r+1)} \frac{\delta^{r(r+1)}}{\delta^{i(i+1)}}} \times \Pi_{r=i+1}^{t-1} e^{-\rho_b^{r(r+1)} \frac{\delta^{r(r+1)}}{\delta^{i(i+1)}}}$$

and

$$e^{\rho_d^{i(i+1)}} = \left(\frac{y_d^t}{y_d^1}\right)^{\frac{1}{\delta^{i(i+1)}\beta_d^{i+1}}} \times \Pi_{r=1}^{i-1} e^{-\rho_d^{r(r+1)} \frac{\delta^{r(r+1)}}{\delta^{i(i+1)}}} \times \Pi_{r=i+1}^{t-1} e^{-\rho_d^{r(r+1)} \frac{\delta^{r(r+1)}}{\delta^{i(i+1)}}}.$$

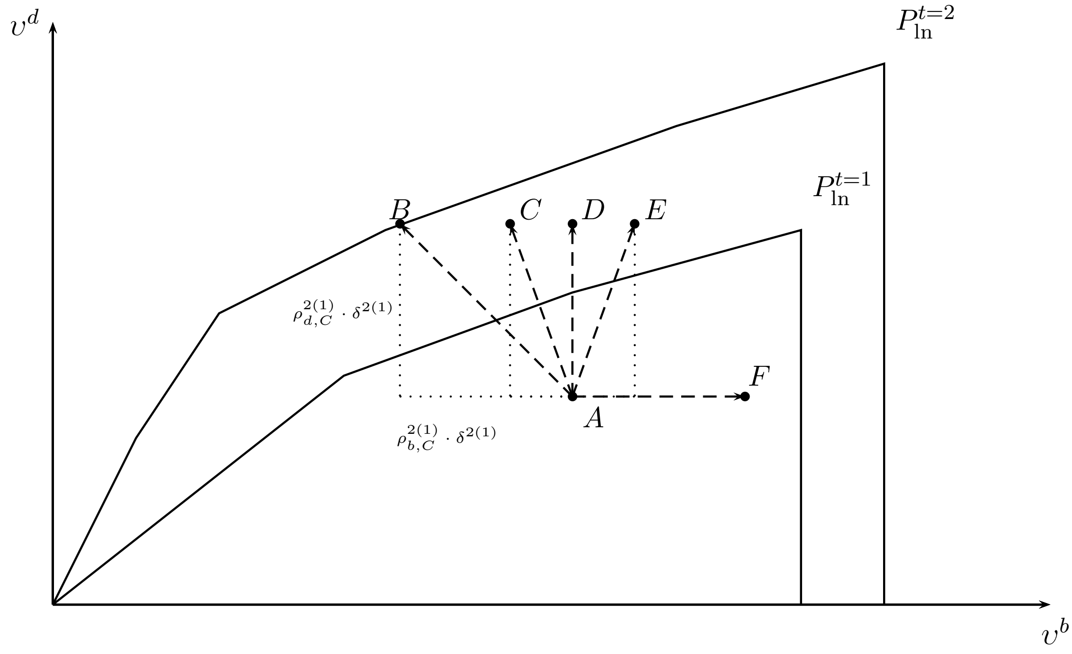

Figure 2: Dynamical deviation ρ in both desirable and undesirable output dimensions

Figure 2 presents an illustration of the dynamical deviation $\rho^{t+1(t)}$ in both dimension, desirable and undesirable. Suppose a firm produces (υ_d, υ_b) outputs represented by the point A at time period $t = 1$. Assume that at time period $t = 2$, several options may occur for the firm :

- Point B corresponds to the case where the firm is efficient at time period $t = 2$ such that $\rho_d^{t+1(t)} = \rho_b^{t+1(t)} = 1$.
- Point C depicts the situation where the firm simultaneously improves and reduces respectively its desirable and undesirable outputs. In such a case, $\rho_d^{t+1(t)} \neq \rho_b^{t+1(t)}$ with $(\rho_d^{t+1(t)}, \rho_b^{t+1(t)}) > 0$.
- Point D describes the circumstance where the firm increases its good outputs and maintains constant its bad outputs with $\rho_d^{t+1(t)} > 0$ and $\rho_b^{t+1(t)} = 0$.
- When the firm increases both desirable and undesirable outputs (point E), $\rho_d^{t+1(t)} \neq \rho_b^{t+1(t)}$ with $\rho_d^{t+1(t)} > 0$ and $\rho_b^{t+1(t)} < 0$.

- Finally, when the firm keeps constant its desirable outputs and increases its bad outputs (point F), then $\rho_d^{t+1(t)} = 0$ and $\rho_b^{t+1(t)} < 0$.

Remark that points C,D,E,F represent the situations where the firm does not operate efficiently at time period $t = 2$. Furthermore, note that there exists reverse situations of points B,C,D,E,F with respect to the signs of parameters $\rho^{t+1(t)}$. Thus, in such a case, the deductions based upon the above explanations are obvious. Evidently, for the cross-time period $t(t+1)$, the same logical reasoning can be applied.

4.4 Exponential Environmental Productivity Index and Indicators in Dynamical Context

Following the notion exposed above, we can state that the EEL can be expressed as below:

$$\begin{aligned} & EEL(x^t, x^{t+1}, y_d^t, y_b^t, y_d^{t+1}, y_b^{t+1}; \gamma^t, \gamma^{t+1}) \qquad (4.9) \\ & = \frac{1}{2}\Big[\big(ED_{\exp^O}^t(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t) - \ln\big((\frac{y_b^1}{y_b^t})^{\frac{1}{\rho_b^{t-1(t)}\beta_b^t}} \times \prod_{r=1}^{t-1} e^{-\frac{\rho_b^r}{\rho_b^{t-1(t)}}\delta^{r(r+1)}}\big)\big) \\ & + \big(\ln\big((\frac{y_b^1}{y_b^t})^{\frac{1}{\rho_b^{t(t-1)}\beta_b^{t-1}}} \times \prod_{r=2}^{t-1} e^{-\frac{\rho_b^r}{\rho_b^{t(t-1)}}\delta^{r(r-1)}}\big) - ED_{\exp^O}^{t+1}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; 0, \beta_d^{t+1}, \beta_b^{t+1})\big)\Big]. \end{aligned}$$

In such case that $\rho^{t-1(t)} = \rho^{t(t-1)} = 1$ then $\delta^{t(t-1)} = \delta^{t-1(t)} = 0$, then the EEL becomes:

$$\begin{aligned} & EEL(x^t, x^{t+1}, y_d^t, y_b^t, y_d^{t+1}, y_b^{t+1}; \gamma^t, \gamma^{t+1}) \\ & = \frac{1}{2}\Big[ED_{\exp^O}^t(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t) - ED_{\exp^O}^{t+1}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; 0, \beta_d^{t+1}, \beta_b^{t+1})\Big]. \end{aligned}$$

In the same argumentation as above, we can express the EML as following:

$$EML(x^t, x^{t+1}, y_d^t, y_d^{t+1}, y_b^t, y_b^{t+1}; 0, \gamma^t, \gamma^{t+1}) = \left[\frac{1 + ED^t_{\exp^O}(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t)}{1 + \ln\left(\left(\frac{y_b^1}{y_b^t}\right)^{\frac{1}{\rho_b^{t-1(t)}\beta_b^t}} \times \prod_{r=1}^{t-1} e^{-\frac{\rho_b^r}{\rho_b^{t-1(t)}}\delta^{r(r+1)}} \right)} \times \frac{1 + \ln\left(\left(\frac{y_b^1}{y_b^t}\right)^{\frac{1}{\rho_b^{t(t-1)}\beta_b^{t-1}}} \times \prod_{r=2}^{t-1} e^{-\frac{\rho_b^r}{\rho_b^{t(t-1)}}\delta^{r(r-1)}} \right)}{1 + ED^{t+1}_{\exp^O}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; 0, \beta_d^{t+1}, \beta_b^{t+1})} \right]^{\frac{1}{2}}. \tag{4.10}$$

Thus, if $\rho^{t-1(t)} = \rho^{t(t-1)} = 1$ meaning that $\delta^{t(t-1)} = \delta^{t-1(t)} = 0$ then:

$$EML(x^t, x^{t+1}, y_d^t, y_d^{t+1}, y_b^t, y_b^{t+1}; 0, \gamma^t, \gamma^{t+1}) = \left[\frac{1 + ED^t_{\exp^O}(x^t, y_d^t, y_b^t; 0, \beta_d^t, \beta_b^t)}{1 + ED^{t+1}_{\exp^O}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; 0, \beta_d^{t+1}, \beta_b^{t+1})} \right]^{\frac{1}{2}}.$$

In the same vein, the dynamical version of the EUCEG is defined by:

$$\begin{aligned}
EUCEG = &\frac{1}{2} \quad [EUCEG^t + EUCEG^{t+1}] \\
= &\frac{1}{2} \quad [(EELO^t + EELO^{t+1}) - (EELI^t - EELI^{t+1})] \\
= &\frac{1}{2} \quad \Bigg[\Big(ED^t_{\exp^O}(x^t, y_d^t, y_b^t; 0, \beta_d^t, 0) - ED^{t+1}_{\exp^O}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; 0, \beta_d^{t+1}, 0) \Big) \\
&+ \left(\left(\frac{y_d^t}{y_d^1}\right)^{\frac{1}{\rho_d^{t(t-1)}\beta_d^{t-1}}} \times \prod_{r=1}^{t-2} e^{-\frac{\rho_d^{r(r-1)}}{\rho_d^{t(t-1)}}\delta_d^{r(r-1)}} - \left(\frac{y_d^t}{y_d^1}\right)^{\frac{1}{\rho_d^{t-1(t)}\beta_d^t}} \times \prod_{r=1}^{t-1} e^{-\frac{\rho_d^{r(r+1)}}{\rho_d^{t-1(t)}}\delta_d^{r(r+1)}} \right) \\
&+ \Big(ED^t_{\exp^O}(x^t, y_d^t, y_b^t; 0, 0, \beta_b^t) - ED^{t+1}_{\exp^O}(x^{t+1}, y_d^{t+1}, y_b^{t+1}; 0, 0, \beta_b^{t+1}) \Big) \\
&+ \left(\left(\frac{y_b^1}{y_b^t}\right)^{\frac{1}{\rho_b^{t(t-1)}\beta_b^{t-1}}} \times \prod_{r=1}^{t-2} e^{-\frac{\rho_b^{r(r-1)}}{\rho_b^{t(t-1)}}\delta_b^{r(r-1)}} - \left(\frac{y_b^1}{y_b^t}\right)^{\frac{1}{\rho_b^{t-1(t)}\beta_b^t}} \times \prod_{r=1}^{t-1} e^{-\frac{\rho_b^{r(r+1)}}{\rho_b^{t-1(t)}}\delta_b^{r(r+1)}} \right) \Bigg].
\end{aligned} \tag{4.11}$$

Note that in the dynamical version of EUCEG $\delta_d \neq \delta_b$ and $\rho_d \neq \rho_b$ in most cases.

4.5 Quantity Index

Take the case of two successive discrete time periods with a single desirable output y_d and a single undesirable output y_b, exposed in the subsection ??. The expression of the distance function was defined as:

$$e^{(\delta^{t(t-1)})} = \begin{cases} \left(\frac{y_b^{t-1}}{y_b^t}\right)^{\frac{1}{\rho_b^{t(t-1)}\beta_b^{t-1}}} \\ \left(\frac{y_d^t}{y_d^{t-1}}\right)^{\frac{1}{\rho_d^{t(t-1)}\beta_d^{t-1}}} \end{cases} \qquad e^{(\delta^{t-1(t)})} = \begin{cases} \left(\frac{y_b^{t-1}}{y_b^t}\right)^{\frac{1}{\rho_b^{t-1(t)}\beta_b^{t}}} \\ \left(\frac{y_d^t}{y_d^{t-1}}\right)^{\frac{1}{\rho_d^{t-1(t)}\beta_d^{t}}} \end{cases}$$

The ratio of these two expressions, with some rearrangements, leads to have the analogous expression of the Törnqvist Total Factor Productivity Index[5] (TTFPI) in the case of a single input and a single output (See and Coelli, 2014). The expression of the TTFPI in multi-input and multi-output context is defined as:

$$TTFPI = \frac{\prod_{k\in[m_d]}\left(\frac{y_{d,k}^t}{y_{d,k}^{t-1}}\right)^{\frac{r_{d,k}^{t-1}+r_{d,k}^t}{2}}}{\prod_{i\in[m_b]}\left(\frac{y_{b,i}^t}{y_{b,i}^{t-1}}\right)^{\frac{s_{b,i}^{t-1}+s_{b,i}^t}{2}}}$$

$s_{b,i}$ and $r_{d,k}$ are respectively the cost and revenue shares of the band and good outputs such that $\sum_{i\in[m_b]} s_{b,i} = 1$ and $\sum_{k\in[m_d]} r_{d,k} = 1$.

Whence, through the formulations of the distance function we have:

$$e^{(2.\delta^{t(t-1)})} = \frac{\left(\frac{y_d^t}{y_d^{t-1}}\right)^{\frac{1}{\rho_d^{t(t-1)}\beta_d^{t-1}}}}{\left(\frac{y_b^t}{y_b^{t-1}}\right)^{\frac{1}{\rho_b^{t(t-1)}\beta_b^{t-1}}}} \quad \text{and} \quad e^{(2.\delta^{t-1(t)})} = \frac{\left(\frac{y_d^t}{y_d^{t-1}}\right)^{\frac{1}{\rho_d^{t-1(t)}\beta_d^{t}}}}{\left(\frac{y_b^t}{y_b^{t-1}}\right)^{\frac{1}{\rho_b^{t-1(t)}\beta_b^{t}}}}. \tag{4.12}$$

[5]Törnqvist Total Factor Productivity measure is multiplicatively complete. O'Donnell (2010, 2012, 2014, 2016) shows that this TFP measure can be decomposed within finer measures of efficiency in terms of aggregate quantities.

And to have the analogous expression to the TTFPI, the ratio becomes:

$$e^{((m_d+m_b).\delta^{t(t-1)})} = \frac{\prod_{k\in[m_d]} \left(\frac{y^t_{d,k}}{y^{t-1}_{d,k}}\right)^{\frac{1}{\rho^{t(t-1)}_{d,k}\beta^{t-1}_{d,k}}}}{\prod_{i\in[m_b]} \left(\frac{y^t_{b,i}}{y^{t-1}_{b,i}}\right)^{\frac{1}{\rho^{t(t-1)}_{b,i}\beta^{t-1}_{b,i}}}} \tag{4.13}$$

and

$$e^{((m_d+m_b).\delta^{t-1(t)})} = \frac{\prod_{k\in[m_d]} \left(\frac{y^t_{d,k}}{y^{t-1}_{d,k}}\right)^{\frac{1}{\rho^{t-1(t)}_{d,k}\beta^{t}_{d,k}}}}{\prod_{i\in[m_b]} \left(\frac{y^t_{b,i}}{y^{t-1}_{b,i}}\right)^{\frac{1}{\rho^{t-1(t)}_{b,i}\beta^{t}_{b,i}}}}. \tag{4.14}$$

It is obvious that in most cases

$$\sum_{k\in[m_d]} \frac{r^{t-1}_{d,k} + r^t_{d,k}}{2} \neq \sum_{k\in[m_d]} \frac{1}{\rho^{t(t-1)}_{d,k}\beta^{t-1}_{d,k}} \quad \text{and} \quad \sum_{i\in[m_b]} \frac{s^{t-1}_{b,i} + s^t_{b,i}}{2} \neq \sum_{i\in[m_b]} \frac{1}{\rho^{t(t-1)}_{b,i}\beta^{t-1}_{b,i}}. \tag{4.15}$$

However, if $\sum_{k\in[m_d]} \rho^{t(t-1)}_{d,k}\beta^{t-1}_{d,k} = \frac{1}{2}$ and $\sum_{i\in[m_b]} \rho^{t(t-1)}_{b,i}\beta^{t-1}_{b,i} = \frac{1}{2}$, then the expression of $e^{((m_d+m_b).\delta^{t(t-1)})}$ can lead to an implicit Törnqvist Total Factor Productivity Index. This TTFPI is qualified to be implicit due to the difference between the weights of the TTFPI based upon market prices (Törnqvist, 1936) and the weights considered above for each component. The same argumentation is made for $e^{((m_d+m_b).\delta^{t-1(t)})}$.

5 Empirical illustration

5.1 Data

The dataset is sourced from many reports and documents of the Ministère de l'écologie, du Développement durable et de l'Énergie (http://www.developpement-durable.gouv.fr). According to the literature review and data availability, one input is selected: Operational costs (Keuros). This input indicator allows to produce different outputs. Thus, one desirable output, Passengers (number of passengers); and one undesirable output represented by the average delay of flights delayed for more than 15 minutes (Delay flights).

Table 1 presents the statistic descriptives of the variables used in this study.

Table 1: Characteristics of inputs and outputs

Variables	Min	Max	Mean	St. Dev.
	Inputs			
Operational costs (Keuros)	17702	1112248	218476.54	349957.95
	Good Output			
Passengers (quantity)	1060705	60970551	12006427.82	16528698.55
	Bad Output			
Delay flights (minutes)	25	51	41.45	4.99

5.2 Results

Under a constant returns-to-scale technology, Table 2 shows the results of productivity indicators and index based on the exponential distance function, for the years 2008 to 2011. As readers can see, the exponential environmental Luenberger productivity indicator presents some infeasabilities (e.g. Nantes and Bordeaux for the consecutive time periods 2009-2010). Since, the exponential Malmquist-Luenberger index and the exponential Luenberger indicator are based upon cross-period efficiency measurement, it is not remarkable that infeasabilities appear for the same Decision Making Units (DMU) at the same cross-period. This is not surprisingly since our new productivity indicator and index inherit the basic structure of classical productivity indicator and index. The special issue of infeasabilities for this kind of productivity index and indicator was already noticed in Briec and Kerstens (2004, 2009). In addition, applying a variable returns-to-scale technology does not avoid infeasabilities occurring in the results. In fact, infeasabilities occur when the efficiency at period t is measured with respect to the production technology set at period $t-1$ or $t+1$ (technological change measurement, see Table 2). Changing returns-to-scale assumption solely impacts the number of efficient DMUs at period t with respect to the technology set at the same time period [6]. This paper focus on the estimation of DMUs' efficiency under a constant returns-to-scale

[6] More DMUs are efficient due to a variable returns-to-scale reference technology

assumption. However, the returns-to-scale measurement of units onto the efficient frontier (projection units) under a variable returns-to-scale reference technology can be further explored (see Krivonozhko et al., 2014). Table 2 also presents the results of the exponential undesirable component environmental generalised productivity indicator. Readers can remark that all infeasabilities are removed through fictive observations constructed by crossing data of two time periods (e.g:($v_d^t, v_b^t + 1$), see Figure 1). Then, a comprehensive environmental productivity analysis can be proposed avoiding infeasabilities.

For the consecutive time periods 2009-2010, the two main airports in France (Paris-CDG and Paris-Orly) presents negative productivity changes (EEL, EML and EUCEG). Regarding the mean of productivity index and indicator over the time period from 2008 to 2011, EEL and EUCEG take negative value and EML takes positive value for both DMUs. Concerning the results of the EUCEG indicator over this cross time period, both airports have negative $EELO$ and positive $EELI$ meaning that these DMUs produce less desirable and more undesirable outputs at year 2010 than 2009. In addition, the EUCEG of these airports takes negative value which involves a downgrading of the ecological productivity.

Table 3 presents the results of dynamical deviations based on the exponential environmental generalized cross-time distance function. Dynamical deviations exposed in Table 3 confirm that Paris-CDG and Paris-Orly present a non-optimal technical efficiency adjustment path. These airports are affected by both internal and external constraints which does not allow the entire diminution of the inefficiency over the period 2008-2010. Furthermore, for Paris-Orly airport, one can see that both internal and external constraints most affect technical efficiency adjustment path in bad output dimension rather than in desirable output dimension (i.e. $-0.4862 < -0.1325 < 1$ for the consecutive time periods 2008-2009).

Table 2: Exponential Productivity Indicators

Exponential Environmental Luenberger

	2008-2009			2009-2010			2010-2011		
	EV	TV	EEL	EV	TV	EEL	EV	TV	EEL
Paris-CDG	0.0225	-0.0516	-0.0290	-0.1020	0.0109	-0.0911	0.1020	-0.0938	0.0082
Paris-Orly	-0.0883	-0.0498	-0.1380	-0.0404	0.0072	-0.0331	0.1722	-0.0904	0.0817
Nice	0.2277	-0.1289	0.0988	0	0.0634	0.0634	-0.0137	-0.2102	-0.2240
Lyon	-0.0245	-0.0607	-0.0852	0.0801	-0.0059	0.0741	0.0920	-0.1320	-0.0401
Marseille	0	-0.0999	-0.0999	0	-0.0116	-0.0116	-0.1391	-0.0640	-0.2032
Toulouse	0.0475	-0.0646	-0.0171	-0.0588	-0.0236	-0.0824	0.1090	-0.0676	0.0415
Bordeaux	0	-0.0988	-0.0988	0	Infinity	Infinity	0	Infinity	Infinity
Ble-Mulhouse	-0.0154	-0.0619	-0.0772	0.0839	-0.0214	0.0624	0.0621	-0.0376	0.0245
Beauvais	0.2019	-0.0637	0.1382	0.0723	-0.0365	0.0358	-0.1886	0.0422	-0.1464
Nantes	0.0431	Infinity	Infinity	-0.0671	Infinity	Infinity	0.0671	Infinity	Infinity
Strasbourg	0.0178	-0.0647	-0.0469	0.0314	-0.0468	-0.0154	0.2020	-0.0303	0.1717

Exponential Malmquist-Luenberger

	2008-2009			2009-2010			2010-2011		
	EV	TV	ML	EV	TV	ML	EV	TV	ML
Paris-CDG	1.0225	0.9503	0.9717	0.9075	1.0103	0.9168	1.1020	0.9146	1.0078
Paris-Orly	0.9307	0.9602	0.8937	0.9693	1.0056	0.9747	1.1507	0.9289	1.0689
Nice	1.2277	0.8905	1.0933	1	1.0676	1.0676	0.9864	0.8028	0.7919
Lyon	0.9810	0.9534	0.9353	1.0664	0.9954	1.0615	1.0825	0.8917	0.9653
Marseille	1	0.9008	0.9008	1	0.9885	0.9885	0.8779	0.9357	0.8214
Toulouse	1.0452	0.9416	0.9842	0.9470	0.9781	0.9263	1.1090	0.9380	1.0402
Bordeaux	1	0.9030	0.9030	1	Infinity	Infinity	1	Infinity	Infinity
Ble-Mulhouse	0.9876	0.9509	0.9391	1.0726	0.9822	1.0536	1.0568	0.9670	1.0220
Beauvais	1.1838	0.9481	1.1223	1.0705	0.9662	1.0343	0.8448	1.0369	0.8759
Nantes	1.0431	Infinity	Infinity	0.9371	Infinity	Infinity	1.0671	Infinity	Infinity
Strasbourg	1.0144	0.9492	0.9629	1.0261	0.9623	0.9875	1.2020	0.9657	1.1608

Exponential Undesirable Component Environmental Generalised

	2008 - 2009			2009 - 2010			2010 - 2011		
	EELO	EELI	EUCEG	EELO	EELI	EUCEG	EELO	EELI	EUCEG
Paris-CDG	-0.0500	0.0247	-0.0747	0.0045	0.1151	-0.1106	0.0471	0	0.0471
Paris-Orly	-0.0430	0.1576	-0.2006	0.0038	0.0408	-0.0370	0.0740	-0.0834	0.1574
Nice	-0.0549	-0.1719	0.1169	-0.0252	-0.1335	0.1084	0.0818	0.3814	-0.2995
Lyon	-0.0254	0.1306	-0.1560	0.0338	-0.1542	0.1880	0.0556	0.0910	-0.0353
Marseille	0.0455	0.2776	-0.2321	0.0313	0.0299	0.0015	-0.0214	0.3655	-0.3868
Toulouse	-0.0107	0	-0.0107	0.0195	0.0247	-0.0052	0.0870	-0.0500	0.1370
Bordeaux	-0.0695	0.0476	-0.1171	0.0981	0.1508	-0.0527	0.1166	-0.2231	0.3397
Ble-Mulhouse	-0.1016	0.0476	-0.1492	0.0699	-0.0723	0.1422	0.2001	0.0488	0.1513
Beauvais	0.0423	-0.0247	0.0669	0.1232	-0.1335	0.2568	0.2267	0.3365	-0.1098
Nantes	-0.0237	0.1178	-0.1415	0.1330	0.1252	0.0079	0.0677	-0.1252	0.1929
Strasbourg	-0.1315	0	-0.1315	-0.0449	0.0488	-0.0937	0.0181	0.0235	-0.0055

Table 3: Dynamical Deviation and Exponential Environmental Generalized Dynamical Luenberger Productivtiy Indicator

$\rho^{t(t+1)}$	**2008(2009)**		**2009(2010)**		**2010(2011)**	
	Bad output	**Good output**	**Bad output**	**Good output**	**Bad output**	**Good output**
Paris-CDG	0.8502	1.7210	-15.4778	0.6030	0	5.6821
Paris-Orly	-1.1560	-0.3150	-0.1450	0.0136	0.3719	0.3300
Nice	1.5733	-0.5028	3.8858	-0.7319	1.5740	-0.3378
Lyon	-0.6472	-0.1256	0.5645	0.1238	-1.4280	0.8731
Marseille	1.9711	-0.3230	2.3834	-2.5011	3.2411	0.1896
Toulouse	0	-0.3265	-1.1006	0.8700	1.0298	1.7914
Bordeaux	0.3767	0.5498	2.5957	-1.6882	Infinity	Infinity
Ble-Mulhouse	-0.2954	-0.6299	0.3334	0.3223	-0.4613	1.8921
Beauvais	0.1034	0.1769	2.3915	2.2072	-10.9341	7.3668
Nantes	2.3263	0.4677	60.4042	-64.1958	Infinity	Infinity
Strasbourg	0	-0.7079	-0.2586	-0.2379	-0.0758	0.0582
$\rho^{t+1(t)}$	**2009(2008)**		**2010(2009)**		**2011(2010)**	
	Bad output	**Good output**	**Bad output**	**Good output**	**Bad output**	**Good output**
Paris-CDG	-0.4792	-0.9700	-1.3121	0.0511	0	0.5013
Paris-Orly	-0.4862	-0.1325	-0.1328	0.0124	0.3581	0.3177
Nice	1.2335	-0.3942	-1.4436	0.2719	-1.9869	0.4264
Lyon	-0.3756	-0.0729	0.7522	0.1650	-0.3858	0.2359
Marseille	-4.7138	0.7725	-2.8009	2.9392	-2.3667	-0.1384
Toulouse	0	-0.0937	-0.1922	0.1519	0.6698	1.1651
Bordeaux	-0.6685	-0.9756	Infinity	Infinity	Infinity	Infinity
Ble-Mulhouse	-0.1586	-0.3382	0.4111	0.3974	-0.4106	1.6841
Beauvais	0.1503	0.2571	2.3619	2.1798	-2.4935	1.6800
Nantes	Infinity	Infinity	Infinity	Infinity	Infinity	Infinity
Strasbourg	0	-0.4423	-0.1945	-0.1789	-0.1393	0.1070

6 Conclusion

The theoretical contribution of this paper consists to propose new exponential environmental productivity index and indicators. These innovative productivity measures are constructed based on an exponential distance function. Looking from a dynamical viewpoint, we define a dynamical version of the exponential distance function allowing to take into account internal and external constraints which affect productivity variation. Innovative exponential generalized dynamical environmental productivity index and indicators are then proposed.

Results of the productivity measures (EEL, EML and EUCEG) based on both consecutive time periods are proposed for a sample of 11 representative French airports. We point out that some infeasabilities can occur when we implement the EEL and the EML which are removed when computing the EUCEG. Such specification allows to define a comprehensive ecological productivity analysis. In addition, for the same sample, we present the "dynamical deviation" which consider both internal and external constraints that may impede optimal technical efficiency variation and technological change achievement in both desirable and undesirable dimensions.

Innovative EUCEG takes the form of an additive complete TFP measure in the sense of O'Donnell (2010, 2012, 2014, 2016) such that a decomposition within finer measures of efficiency in terms of aggregate quantities of the EUCEG could be an extension of this paper.

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Appendix A

Proof of Proposition ??:

Since $ED^{t(t-1)}_{\ln^O}(u^{t-1}, v_b^{t-1}, v_d^{t-1}; 0, \beta_b^{t-1}, \beta_d^{t-1}) = \delta^{t(t-1)}$, the production units at time $\tau = t$ can be expressed as:

$$(u^t, v_b^t, v_d^t) = (u^{t-1}, v_b^{t-1}, v_d^{t-1}) + ED^{t(t-1)}_{\ln^O}(u^{t-1}, v_b^{t-1}, v_d^{t-1}; 0, \beta_b^{t-1}, \beta_d^{t-1}) \cdot (0, \beta_b^{t-1}, \beta_d^{t-1}).$$

Suppose that the firm fits its production units at a proportion $\rho^{t(t-1)} \in \mathbb{R}$ of the distance function such that (u^t, v_b^t, v_d^t) can be inefficient and $\delta^t \geq 0$. Thus, each production unit can be expressed as:

$$\begin{cases} u^t &= u^{t-1} \\ v_b^t &= v_b^{t-1} - \rho_b^{t(t-1)} \delta^{t(t-1)} \beta_b^{t-1} \\ v_d^t &= v_d^{t-1} + \rho_d^{t(t-1)} \delta^{t(t-1)} \beta_d^{t-1} \end{cases}$$

Whence, in such case, the neperian distance function can be write as follows:

$$\begin{aligned} \delta^{t(t-1)} &= \frac{v_b^{t-1} - v_b^t}{\rho_b^{t(t-1)} \beta_b^{t-1}} &= \frac{\ln(y_b^{t-1}) - \ln(y_b^t)}{\rho_b^{t(t-1)} \beta_b^{t-1}} \\ &= \frac{v_d^t - v_d^{t-1}}{\rho_d^{t(t-1)} \beta_d^{t-1}} &= \frac{\ln(y_d^t) - \ln(y_d^{t-1})}{\rho_d^{t(t-1)} \beta_d^{t-1}}. \end{aligned}$$

The exponential formulation is:

$$\begin{aligned} \exp(\delta^{t(t-1)}) &= \exp\left(\frac{\ln(y_b^{t-1}) - \ln(y_b^t)}{\rho_b^{t(t-1)} \beta_b^{t-1}}\right) \\ &= \exp\left(\frac{\ln(y_d^t) - \ln(y_d^{t-1})}{\rho_d^{t(t-1)} \beta_d^{t-1}}\right). \end{aligned}$$

Then,

$$e^{(\delta^{t(t-1)})} = \left(\frac{y_b^{t-1}}{y_b^t}\right)^{\frac{1}{\rho_b^{t(t-1)}\beta_b^{t-1}}} = \left(\frac{y_d^t}{y_d^{t-1}}\right)^{\frac{1}{\rho_d^{t(t-1)}\beta_d^{t-1}}}.$$

In such case that the firm is efficient at $\tau = t$ so $\rho^{t(t-1)} = 1$, the distance function becomes:

$$e^{(\delta^{t(t-1)})} = \left(\frac{y_b^{t-1}}{y_b^t}\right)^{\frac{1}{\beta_b^{t-1}}} = \left(\frac{y_d^t}{y_d^{t-1}}\right)^{\frac{1}{\beta_d^{t-1}}}.$$

Proof of Proposition ??:

Assume that the firm is inefficient at the $\tau = t$ such that $\rho^{t-1(t)} \neq 1$, then:

$$\begin{cases} u^{t-1} & = u^t \\ v_b^{t-1} & = v_b^t + \rho_b^{t-1(t)}\delta^{t-1(t)}\beta_b^t \\ v_d^{t-1} & = v_d^t - \rho_d^{t-1(t)}\delta^{t-1(t)}\beta_d^t \end{cases}$$

Hence the distance function can be expressed as follows:

$$\begin{aligned} \delta^{t-1(t)} &= \tfrac{v_b^{t-1}-v_b^t}{\rho_b^{t-1(t)}\beta_b^t} &= \tfrac{\ln(y_b^{t-1})-\ln(y_b^t)}{\rho_b^{t-1(t)}\beta_b^t} \\ &= \tfrac{v_d^t-v_d^{t-1}}{\rho_d^{t-1(t)}\beta_d^t} &= \tfrac{\ln(y_d^t)-\ln(y_d^{t-1})}{\rho_d^{t-1(t)}\beta_d^t}. \end{aligned}$$

The exponential formulation is:

$$\begin{aligned} \exp(\delta^{t-1(t)}) &= \exp\left(\frac{\ln(y_b^{t-1}) - \ln(y_b^t)}{\rho_b^{t-1(t)}\beta_b^t}\right) \\ &= \exp\left(\frac{\ln(y_d^t) - \ln(y_d^{t-1})}{\rho_d^{t-1(t)}\beta_d^t}\right). \end{aligned}$$

Then,

$$e^{\delta^{t-1(t)}} = \left(\frac{y_b^{t-1}}{y_b^t}\right)^{\frac{1}{\rho_b^{t-1(t)}\beta_b^t}} = \left(\frac{y_d^t}{y_d^{t-1}}\right)^{\frac{1}{\rho_d^{t-1(t)}\beta_d^t}}.$$

If $\rho^{t-1(t)} = 1$, the notion set above becomes:

$$e^{\delta^{t-1(t)}} = \left(\frac{y_b^{t-1}}{y_b^t}\right)^{\frac{1}{\beta_b^t}} = \left(\frac{y_d^t}{y_d^{t-1}}\right)^{\frac{1}{\beta_d^t}}.$$

Appendix B

Proof of Proposition ??: In the case where the firm is inefficient then $\rho^{t(t-1)} \neq 1$ at $\tau = t$, the decomposition of the production units is:

$$\begin{cases} u^t &= u^1 \\ v_b^t &= v_b^1 - \rho_b^{2(1)}\delta^{2(1)}\beta_b^1 - \rho_b^{3(2)}\delta^{3(2)}\beta_b^2 - ... - \rho_b^{t(t-1)}\delta^{t(t-1)}\beta_t^{t-1} \\ v_d^t &= v_d^1 + \rho_d^{2(1)}\delta^{2(1)}\beta_d^1 + \rho_d^{3(2)}\delta^{3(2)}\beta_d^2 + ... + \rho_d^{t(t-1)}\delta^{t(t-1)}\beta_d^{t-1} \end{cases}$$

Thus,

$$\begin{cases} u^t &= u^1 \\ v_b^t &= v_b^1 - \beta_b \sum_{r=2}^{t-1} \rho_b^{r(r-1)}\delta^{r(r-1)} - \rho_b^{t(t-1)}\delta^{t(t-1)}\beta_b^{t-1} \\ v_d^t &= v_d^1 + \beta_d \sum_{r=2}^{t-1} \rho_d^{r(r-1)}\delta^{r(r-1)} + \rho_d^{t(t-1)}\delta^{t(t-1)}\beta_d^{t-1} \end{cases}$$

Hence, the distance function in dynamical context is:

$$\begin{aligned} \delta^{t(t-1)} &= \frac{v_b^1 - v_b^t - \beta_b \sum_{r=2}^{t-1} \rho_b^{r(r-1)}\delta^{r(r-1)}}{\rho_b^{t(t-1)}\beta_b^{t-1}} &= \left(\frac{v_b^1 - v_b^t}{\rho_b^{t(t-1)}\beta_b^{t-1}}\right) - \frac{1}{\rho_b^{t(t-1)}} \sum_{r=2}^{t-1} \rho_b^{r(r-1)}\delta^{r(r-1)} \\ &= \frac{v_d^t - v_d^1 - \beta_d \sum_{r=2}^{t-1} \rho_d^{r(r-1)}\delta^{r(r-1)}}{\rho_d^{t(t-1)}\beta_d^{t-1}} &= \left(\frac{v_d^t - v_d^1}{\rho_d^{t(t-1)}\beta_d^{t-1}}\right) - \frac{1}{\rho_d^{t(t-1)}} \sum_{r=2}^{t-1} \rho_d^{r(r-1)}\delta^{r(r-1)}. \end{aligned}$$

Whence,

$$\begin{aligned} \delta^{t(t-1)} &= \left(\frac{\ln(y_b^1) - \ln(y_b^t)}{\rho_b^{t(t-1)}\beta_b^{t-1}}\right) - \frac{1}{\rho_b^{t(t-1)}} \sum_{r=2}^{t-1} \rho_b^{r(r-1)}\delta^{r(r-1)} \\ &= \left(\frac{\ln(y_d^t) - \ln(y_d^1)}{\rho_d^{t(t-1)}\beta_d^{t-1}}\right) - \frac{1}{\rho_d^{t(t-1)}} \sum_{r=2}^{t-2} \rho_d^{r(r-1)}\delta^{r(r-1)}, \end{aligned}$$

then the exponential expression is as follows:

$$\exp(\delta^{t(t-1)}) = \exp\Big(\big(\frac{\ln(y_b^1)-\ln(y_b^t)}{\rho_b^{t(t-1)}\beta_b^{t-1}}\big) - \frac{1}{\rho_b^{t(t-1)}}\sum_{r=2}^{t-1}\rho_b^{r(r-1)}\delta^{r(r-1)}\Big)$$
$$= exp\Big(\big(\frac{\ln(y_d^t)-\ln(y_d^1)}{\rho_d^{t(t-1)}\beta_d^{t-1}}\big) - \frac{1}{\rho_d^{t(t-1)}}\sum_{r=2}^{t-1}\rho_d^{r(r-1)}\delta^{r(r-1)}\Big).$$

Thus,

$$e^{(\delta^{t(t-1)})} = \Big(\frac{y_b^1}{y_b^t}\Big)^{\frac{1}{\rho_b^{t(t-1)}\beta_b^{t-1}}} \times \prod_{r=2}^{t-1} e^{-\frac{1}{\rho_b^{t(t-1)}}\rho_b^{r(r-1)}\delta^{r(r-1)}}$$
$$= \Big(\frac{y_d^t}{y_d^1}\Big)^{\frac{1}{\rho_d^{t(t-1)}\beta_d^{t-1}}} \times \prod_{r=1}^{t-2} e^{-\frac{1}{\rho_d^{t(t-1)}}\rho_d^{r(r-1)}\delta^{r(r-1)}}.$$

In such case that the firm is efficient at $\tau = t$ so $\rho^{t(t-1)} = 1$, the distance function becomes:

$$e^{(\delta^{t(t-1)})} = \Big(\frac{y_b^1}{y_b^t}\Big)^{\frac{1}{\beta_b^{t-1}}} \times \prod_{r=2}^{t-1} e^{-\rho_b^{r(r-1)}\delta^{r(r-1)}} = \Big(\frac{y_d^t}{y_d^1}\Big)^{\frac{1}{\beta_d^{t-1}}} \times \prod_{r=2}^{t-1} e^{-\rho_d^{r(r-1)}\delta^{r(r-1)}}.$$

Proof of Proposition ??: Assume that the firm is inefficient at the time $\tau = t$ such that $\rho^{t-1(t)} \neq 1$, then

$$\begin{cases} u^1 &= u^t \\ v_b^1 &= v_b^t + \rho_b^{1(2)}\delta^{1(2)}\beta_b^2 + \rho_b^{2(3)}\delta^{2(3)}\beta_b^3 + ... + \rho_b^{t-1(t)}\delta^{t-1(t)}\beta_b^t \\ v_d^1 &= v_d^t - \rho_d^{1(2)}\delta^{1(2)}\beta_d^2 - \rho_d^{2(3)}\delta^{2(3)}\beta_d^3 - ... - \rho_d^{t-1(t)}\delta^{t-1(t)}\beta_d^t \end{cases}$$

Hence the distance function can be expressed as follows:

$$\delta^{t-1(t)} = \frac{v_b^1 - v_b^t}{\rho_b^{t-1(t)}\beta_b^t} - \frac{1}{\rho_b^{t-1(t)}}\sum_{r=1}^{t-1}\delta^{r(r+1)}\rho_b^{r(r+1)}\beta_b^{r+1}$$
$$= \frac{v_d^t - v_d^1}{\rho_d^{t-1(t)}\beta_d^t} - \frac{1}{\rho_d^{t-1(t)}}\sum_{r=1}^{t-1}\delta^{r(r+1)}\rho_d^{r(r+1)}\beta_d^{r+1}.$$

Thus, the exponential formulation of the distance function is:

$$\begin{aligned} e^{\delta^{t-1(t)}} &= \left(\frac{y_b^1}{y_b^t}\right)^{\frac{1}{\rho_b^{t-1(t)}\beta_b^t}} \times \prod_{r=1}^{t-1} e^{-\frac{\rho_b^{r(r+1)}}{\rho_b^{t-1(t)}}\delta^{r(r+1)}} \\ &= \left(\frac{y_d^t}{y_d^1}\right)^{\frac{1}{\rho_d^{t-1(t)}\beta_d^t}} \times \prod_{r=1}^{t-1} e^{-\frac{\rho_d^{r(r+1)}}{\rho_d^{t-1(t)}}\delta^{r(r+1)}}. \end{aligned}$$

If $\rho^{t-1(t)} = 1$, the notion set above becomes:

$$e^{\delta^{t-1(t)}} = \left(\frac{y_b^1}{y_b^t}\right)^{\frac{1}{\beta_b^t}} \times \prod_{r=1}^{t-1} e^{-\rho_b^{r(r+1)}\delta^{r(r+1)}} = \left(\frac{y_d^t}{y_d^1}\right)^{\frac{1}{\beta_d^t}} \times \prod_{r=1}^{t-1} e^{-\rho_d^{r(r+1)}\delta^{r(r+1)}}.$$