Fishery Management with Poorly Known Dynamics

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Abstract

Adapting fishing capacity to a highly variable environment remains a complex challenge for managers who have to deal with non-linear dynamics of fish population and harvest levels. In this paper a recent method of stochastic control is adapted to deal with a general fishery management problem under multiples sources of uncertainty. The question is about adjusting permanently the management rule or to hold a fixed policy avoiding additional noise. The mathematical problem developed here, though oversimplified, represents an original approach to the fishery management issue inspired by the monetary policy challenge of a central bank (Brainard principle). It assumes that Control Variation Increases the level of Uncertainty (namely CVIU approach) under particular conditions, resulting in preferable inaction regions for managers. We specify these conditions to show that the management of a poorly known fishery is still possible by using a CVIU approach.

Keywords: Fishery Management, Optimal Control, Dynamics Uncertainty, CVIU

1 Introduction

Overexploitation of common renewable resources like fisheries may turn to a situation where it becomes optimal to overexploit fisheries to extinction [1]. Such cases occur when the pace of biomass growth is too slow, property rights are not well defined, fishing costs are too small or hidden by subsidies, discount rates and fish prices are too high, and technical change make fisheries still profitable at low levels of biomass ([2], [3], [4], [5]). The situation worsens with uncertainty if the target of sustainable yield is overestimated by scientists or set above the advised level for political or short-sighted economic reasons ([6], [7]).

Sources of uncertainty in the management of renewable resources are multiple. Several surveys and classifications of uncertainty are proposed by authors ([8], [9], [10], [11]). Uncertainty in fisheries may

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come from random fluctuations influenced by environmental variability, wrong parameter estimates leading to stock assessment errors, and structural uncertainty stemming from a lack of knowledge about the nature of the fishery system and an inaccurate implementation of harvest quotas ([12], [13], [14], [9], [11]). These distinct sources of uncertainty may have different implications for managers. If environmental variability is unpredictable because of a random multiplicative shock disturbing the growth but the stock is correctly assessed every period, then managers can choose a constant optimal escapement rule avoiding stock exhaustion [12]. However, constant escapement is no longer possible when managers do not know accurately what the stock is in the current period [13]. As a result, the optimal management policy can be significantly different, justifying less cautious strategies and even extinction in some rare cases [15]. By introducing stochastic behaviors of fishers when choosing their catch levels on the basis of managers decision, the outcome becomes even more unpredictable and analytical solutions can hardly be discovered [14].

Uncertainty is likely to increase by considering the marine ecosystems and their dynamics. Beyond the errors of parameter estimates, the uncertainty about model structures has been poorly regarded by scientists. All ecosystem dynamics models simplify the structure of the food web, the nature of ecological interactions, and the demographic structure of populations [11], making predictions highly risky. Such a complexity has resulted in the development of new management policies called adaptive (or experimental) management, robust management, or balanced harvest, sometimes advocating opposite measures to what had been advised so far by scientists to managers, such as increasing selectivity ([16], [17]). In this ecosystem approach of fisheries, what matters most is the conservation of ecosystem structure and functioning in order to maintain ecosystem services and marine biodiversity. By broadening the range of sizes and species being caught in fisheries, the balanced harvest policy would provide economic benefits derived from food security, higher productivity of the ecosystem and greater resilience to climate change [18]. It has even been reported for some African lake fisheries that no management at all could bring better outcomes than a strictly selective management, by conserving the structure of the ecosystem [19]. This option is challenged by other authors who consider that balanced harvesting should not mean a laissez-faire approach and the rejection of selective management, but rather upscale selectivity to the integrated ecosystem level ([17], [20]).

Questions about the timely type of management is more than ever on top of the research agenda, because management decisions may directly influence the level of uncertainty surrounding the fishery. Taking this debate to the extreme, uncertainty can lead to two opposite options for managers: doing nothing (or keeping the rule unchanged whatever the circumstances), well known in the monetary economic literature as the Brainard principle [21], or acting as vividly and often as possible to adapt decisions to the best existing knowledge of biomass, effort and harvest targets. The second strategy can be the optimal one in case of necessary learning process [22] but both solutions can also be deemed sub-optimal or unsatisfying solutions on both biological and economic grounds if the biomass dynamics is affected by inappropriate management decisions. Indeed, social and natural systems may respond to management options in multiple unexpected ways. Despite its influence, the human behavior is much less studied than any other source of uncertainty [23]. However, investment decisions in fisheries, like in other economic activities, must rely on a certain amount of stability depending on the state of the resource stock, the total allowable catch (TAC) decided by managers, the fishers individual catch share, etc. Fishers behavior can become erratic if their regulated access to resources varies from period to period because of adaptive management. Within the framework of the Common Fisheries Policy, TACs and quotas for most stocks are set annually in December by the Council of fisheries ministers, creating an uncertain climate for European fishers whose short and mid-term business depends on such decisions. Most producers organizations claim for multi-annual quotas to secure their own activity, rather than operating in a context of incomplete information [24].

Natural systems are also profoundly affected by fishing [25]. Engwerda [26] analyzed the level of activities of fishers in an uncertain world where fishers may take care —or not— of a biomass stabilization constraint. As usual in robust control models, the growth of fish stocks (e.g. the state variable) depends on some aggregate uncertain factors (water quality, weather, etc.). This disturbance is somehow independent from any stock level or any amount of fishing effort. However, the way fish is caught may easily put stock growth onto non-linear dynamics ([27], [25], [28]). As reported by FAO (2003), fishing impacts natural stocks by reducing population abundance, its spawning capacity and, potentially, the population growth parameters (growth, fecundity, etc.) [29]. Fishing modifies the age and size structure, sex ratio, genetics and species composition of the targeted renewable resources, as well as their associated and dependent species [30]. For example, bottom trawling is a fishing method that drags a large net across the seafloor, destroying some essential habitats for marine fauna [31]. Large amounts of bycatch are unintentionally caught and thrown overboard dead or dying, changing the community structure and trophic interactions [30]. All these negative externalities can obviously reduce the carrying capacity of the whole ecosystem. Anderson [27] studied larval fish records over fifty years in California and showed that the intrinsic growth rate of the population had increased because of fishing, amplifying the non-linear dynamics of the biomass. This evidence may appear counter-intuitive because the truncation of the stock across the age structure cuts down the number of highly productive big old fat fecond female fish (BOFFFFs) [32], and therefore should reduce population growth. However, this truncation produces other interesting dynamic effects. Higher survival rates of offspring individuals compensate their shorter life history traits. Fewer competitors for food and cannibalism enhance the somatic growth and the per-capita fecundity of the population, therefore increasing the intrinsic growth rate which amplifies the stock fluctuations. It is well known that unstable dynamics may result from increased values of intrinsic growth rates, but fairly high values of this rate are required for a single species stock. However, a truncated age-size structure of a population interacting with other related species can create unpredictable variability with only a small increase of the growth rate parameter [27]. The resulting non-linear dynamics urges the need for a precautionary approach of fishery management.

In this article, we assume that the level of uncertainty is directly linked, and somehow proportional, to the evolution of the gap between the current fish stock and a desired one which can be a biological optimized stock such as the maximum sustainable yield (MSY). Moving away from this optimal level will create more uncertainty, thus affecting the growth function parameters and the biomass dynamics. We also assume that catch levels influence the amount of uncertainty, either because the ecosystem and the carrying capacity are modified [30], or because it endogenously changes the intrinsic growth rate [27]. As recommended by FAO (2017), it becomes urgent to move away before being trapped in a situation where only extreme strategies are possible (stock exhaustion, area closure, fishing ban). The central issue of the present research can be summarised as follows: what are the conditions under which changing the management rules creates more uncertainty and unpredictable outcomes than addressing the overexploitation problem in a context of poor knowledge about fish stocks, harvest levels and population dynamics? We expect situations where taking no other action than sticking to the long-term management rule (e.g. a fixed Total Allowable Cach) is more appropriate for both natural and social systems than adjusting permanently the amount of authorized catches on the basis of new but inconsistent knowledge. To answer these questions, we propose to use a framework analyzing the conditions under which the value of the Control Variation Increases the level of Uncertainty (CVIU). Namely, this CVIU approach will allow to better understand resource management in a context of poorly known dynamics (see [33],[34], [35]).

The rest of this paper is organized as follows. Section 2 presents the bioeconomic model and how uncertainty affects the parameters of the model. Section 3 presents the CVIU optimization problem. Section 4 provides the CVIU solutions and shows the existence of the Inaction Region. Section 5 discusses the characteristics of the inaction region in the bioeconomic model, and provides numerical simulations. As demonstrated, if state uncertainty (e.g. surrounding the dynamics of biomass) and the discount factor are key elements of the existence of the inaction region, control uncertainty (e.g. the impact of the TAC policy) explains the location of the inaction region around the desired state level. Section 6 provides some concluding remarks and policy implications.

2 Bio-Economics of the fisheries and the CVIU approach

2.1 Economic consideration

The fish population at any time t,$\{z(t)\}_{t\geq 0}$, is harvested under a controlled rate $\{h(t)\}_{t\geq 0}$. We assume that the fishers will always harvest at their technological capacity and according to the level set by the Total Allowable Catch (TAC) instrument.

$$h(t) = \text{TAC}(t) = h_e + u(t) \tag{1}$$

Eqn, (1) means that fishers cannot catch more than the TAC at time t. It means also that the TAC is split into two components: a fixed element, h_e, and a flexible element, $u(t)$. Consequently, $u(t)$ can be interpreted as the incremental, $u(t) > 0$, or reduced, $u(t) < 0$, amount of allowed catches around the fixed TAC. Such a flexibility will allow the policymaker to adjust, if necessary, the TAC level at every period t in order to align increasing profits of fishers and enhancing the biomass level, given the uncertainty problem.

Let assume that fishers are identical and small (e.g. price takers). Let also assume that, at the industrial level, the individual total costs and revenues of fishers can be summed up to give the following overall profit function for the fishing sector:

$$\pi(h(t)) = ph(t) - ch(t)^2 \tag{2}$$

where quadratic costs are used to express decreasing returns to scale, and with a constant market price p. Without dynamic consideration, maximization of (2) leads to the static optimal solution $h^*(t) = p/2c$.

Given (1), and rewriting the profit as a function of $u(t)$, we have the profit definition based on the value of the the managers flexible instrument, that is:

$$\pi(h_e, u_t) = (ph_e - ch_e^2) + \big((p - 2ch_e)u(t) - cu(t)^2\big) \equiv \overline{\pi} + \pi(u(t))) \tag{3}$$

This instantaneous profit is made up with two components: a maximum fixed value, defined by $\overline{\pi} = ph_e - ch_e^2$, which depends on h_e, the fixed component of the TAC value, and, a variable value, $\pi(u(t))$, relying on the amount by which the policymaker restricts, $u(t) < 0$, or increases, $u(t) > 0$, catches below, or over, the fixed component of the TAC at time t.

We assume that the government wants to minimize the following expected cost function (in the case of constant price) by setting an optimal sequence $\{u(t)\}, t \geq 0$,

$$E\Big[\int_0^T e^{-\alpha t}\left(-\pi(h_e, u_t) + x(t)^2 + qx(t)\right) dt\Big] = E\Big[\int_0^T e^{-\alpha t}\left(-\pi(u(t)) + x(t)^2 + qx(t) - \overline{\pi}\right) dt\Big] \tag{4}$$

with $x(t)$ the gap value between a desired biomass level and the current stock level, and with α a discount rate.

Given (3), one can rewrite this minimization problem in a standard linear quadratic setting such as:

$$J(s, x, u(\cdot)) = E^x\left[\int_s^\infty e^{-\alpha t}\left(Qx(t)^2 + qx(t) + Ru(t)^2 + ru(t) - K\right) dt\right], \tag{5a}$$

with $Q = 1$, $r = -(p - 2ch_e)$, $R = c > 0$, $K = \overline{\pi} > 0$, and $q < 0$ ($q > 0$) which means a reward (penalty) for achieving a greater biomass level.

By defining $\{u(t)\}_{t\geq 0}$ it is possible to offset (if $u(t) \neq 0$) on the agreed nominal harvesting rate h_e. In other words, the manager has to trade off between maximizing a profit level and keeping $x(t)$ closed to zero, i.e. keeping the biomass level at its desired level. The term $qx(t)$ is added and produce one of the two impacts. If $q < 0$, it means that when a biomass is over the desired level, it can provide a small reward by itself. On the contrary, if $q > 0$, a biomass over the desired level is seen as a cost.

2.2 Biological consideration

The fishery manager wishes to control, via authorized harvesting, the dynamics of stocks for some species of economic interest, striving to avoid depletion or even exhaustion of the population. Assume that the population follows a natural growth behavior subject to small fluctuations of its growth rate, that can be well represented by additive noise with variable intensity. However, because of the complexity of ecosystems, the actual parameters defining the growth curve are not known, and only approximated estimates of the parameters can be used. This assumption is quite realistic since these values are highly influenced by the fluctuations of the environment due to climate changes, new patterns of migration, interactions with other species, etc. Such uncertainty should be taken into account, including its relationship with either the intensity of the control or the evolution of the state, both having to be properly defined in the model.

Typically, when modeling the population dynamics, the growth of the biomass is expressed as a function of the biomass itself, $z(t)$, and as a function of the aggregated catch level, $h(t)$. Among the potential functional candidates, the logistic growth function is frequently used by fisheries economists. This logistic function will lead to an inverted U-shaped curve at a given level of biomass or effort. It is well known that the maximum sustainable yield (MSY) is reached when the growth (e.g. the productivity of the biomass) is at its maximum. It is also well known that the biological equilibrium may not correspond to an optimal economic solution when taking into consideration other parameters such as prices, costs, technological change, discount rate, and the access regime to resources ([3], [4], [5]).

Interestingly, both parameters of the logistic curve, the *carrying capacity* and the *intrinsic growth rate* r, may not be perfectly known. At least three sources of uncertainty can be defined as a function of some noise ϵ. Firstly, the policymaker may not know what is the exact current biomass, $z_t + \epsilon_x$. Secondly, the intrinsic growth rate can be either underestimated or overestimated, $r + \epsilon_r$. Thirdly, one may ignore the accurate carrying capacity, $K + \epsilon_K$. Traditionally, these noises are set exogenously. In this article we will consider that uncertainties about the intrinsic growth rate and the carrying capacity are functions of the gap between the observed biomass and the desired one, and of the value of the control in use.

In order to discuss this last assumption, along with a nonlinear growth curve, we know that, at a given point of interest, one can locally approximate the nonlinear dynamics by a linear one. The dynamics may locally evolve according to:

$$\dot{z}(t) = az(t) + b - h(t) \tag{6}$$

Because of uncertainty in the logistic curve, the linear equation may exhibit the following parameter uncertainties

$$\dot{z} = (a + \epsilon_z)z(t) + (b + \epsilon_b) - h(t) = az(t) + b + (a\epsilon_z z(t) + \epsilon_b) - h(t) \tag{7}$$

In Eq. (7), a direct link between the value of the state and the degree of uncertainty is considered.

For the logistic model,

$$a = r\left(1 - \frac{2z_e}{K}\right), \qquad b = r\left(1 - \frac{2z_e}{K}\right) z_e \tag{8}$$

Another source of uncertainty may come from the harvesting impact. By fishing "big fish" rather than small ones, and because fishing gears can somehow degrade the ecosystem, either the carrying capacity (K) or the growth rate can be negatively impacted (Lindholm et al. 1999, Anderson et al. 2008). Including this harmful effect in equation (6) will give:

$$\dot{z}(t) = az(t) + b - (1 + \varepsilon_h)h(t) = az(t) + b - h(t) - \varepsilon_h h(t) \tag{9}$$

Which means again that the net impact of fishing is more than one-to-one.

Taking into account all uncertainties by aggregating (7) and (9), and adding another source of uncertainty due to external and unpredictable events (such as climate oscillations), the general bio-economic dynamics equation becomes:

$$\dot{z} = az(t) + b - h(t) + (a\epsilon_z z(t) + \epsilon_b - \epsilon_h h(t)) + \epsilon \tag{10}$$

Such a dynamic equation means that we poorly understand the population dynamics. It means also that, ceteris paribus, there exists a link between the state (biomass) level, $z(t)$, the action value, $h(t)$, and the degree of uncertainty. Finally, and from the managers perspective, managing uncertainty is likely to be more important when the biomass value is lower than the targeted value z_e, than on the other side, the surplus side. The system is actually more dangerously uncertain on the shortage side because, for example, the whole ecosystem may be affected by some irreversible damages. In the next section, we show that the management of a poorly known fishery system, with the described features, is still made possible by using a CVIU approach [35] based on the control of an associate stochastic differential equations (SDE).

3 The CVIU for Growth Modeling

3.1 CVIU state dynamics

With a fish population being harvested, we assume that the amount of biomass $\{z(t)\}_{t\geq 0}$ evolves according to the following SDE,

$$dz(t) = G(z(t)) - h(t)dt + \sigma dW(t), \quad t \geq 0, \tag{11}$$

$z(0) = z > 0$ is the initial biomass, σ is the diffusion coefficient and $\{W(t)\}_{t\geq 0}$ is the standard one-dimensional Brownian Motion. In the model the latter stands for natural disturbances in the population growth, whereas $\{h(t)\}_{t\geq 0}$ is the harvesting rate, a resulting effect of the economic motivation and of the policy that has being imposed, which are assumed to coincide, from the fact that the TAC performs as an imposed limiting regulation. Whenever harvest exceeds natural growth, the population level drifts to decline, and conversely.

Let us define $x(t) = z(t) - z_e$, where z_e is a desired biomass level (for example one may wish to set $z_e = z_{\text{MSY}}$). Near an equilibrium point (z_e, h_e), namely $h_e = G(z_e)$, around which the process gravitates if it is stable equilibrium, a CVIU model can be used to represent (11).

Consider that a system governed by (11) is operating near an equilibrium point (z_e, h_e) and for control purposes, a linear model of its dynamics is adopted to that point, by rewriting the original state and control processes in (11) with a simple change of variables: $x(t) := z(t) - z_e$ and $u(t) := h(t) - h_e$. Assume that G is differentiable at (z_e, h_e) in such a way that

$$\begin{aligned} dx(t) = dz(t) &= G\left(z_e + x(t), h_e + u(t)\right) dt + \sigma dW(t) \\ &\cong \left(A^0 x(t) + B^0 u(t)\right) dt + \sigma dW(t), \quad t \geq 0, \end{aligned} \tag{12}$$

where $x = \{x(t)\}_{t\geq 0}$ and $u = \{u(t)\}_{t\geq 0}$ now describe the state and the control variations, respectively, and the *exact local model* of the original process employs the matrices A^0 and B^0 of compatible dimensions. For example, in the case of logistic growth, one has that $A^0 = a, h_e = b$ in (8) and $B^0 = -1$.

As already stressed, (11) might be a poorly known system overall, one that cannot be probed and that are few historical data to learn on a formal statistical procedure. No more than a rough model is available, say a continuous function $\tilde{G} : \mathbb{R} \times \mathbb{U} \to \mathbb{R}^n$, such that $\tilde{G}(z_e, h_e) = G(z_e, h_e)$ and that is differentiable at (z_e, h_e), with parameters A and B, possibly with $A \neq A^0$ and $B \neq B^0$. Note that the errors introduced by the rough model $\tilde{G}$ in regards to what would be the exact description of the system G, are bound to arise even in the first order terms. The idea is to handle these model uncertainties, by adding extra noise terms to the linear (A, B) model.

For the simplest scalar case, we propose to replace the linearized model in (12) by the following version of asymmetric CVIU model:

$$
\begin{aligned}
dx(t) = (Ax(t) + Bu(t))\, dt + \sigma dW(t) + (\bar{\sigma}_x + (\sigma_x^+ x(t)^+ - \sigma_x^- x(t)^-)dW^x(t) \\
+ (\bar{\sigma}_u + (\sigma_u^+ u(t)^+ - \sigma_u^- u(t)^-)dW^u(t),
\end{aligned} \tag{13}
$$

$t \geq 0$, where,

$$
\begin{aligned}
x(t)^+ &= \max(x(t), 0), \quad x(t)^- = \min(x(t), 0) \\
u(t)^+ &= \max(u(t), 0), \quad u(t)^- = \min(u(t), 0),
\end{aligned} \tag{14}
$$

A and B are scalars and W^x, W^u are added one-dimensional standard BM each. The two groups of scalars $\bar{\sigma}_x, \sigma_x^+, \sigma_x^-$ and $\bar{\sigma}_u, \sigma_u^+, \sigma_u^-$ possesses the same signal and without loss we assume that they are all nonnegative.

If $\sigma_x^+ = \sigma_x^-$ and $\sigma_u^+ = \sigma_u^-$ one can write $\sigma_x^+ x^+ - \sigma_x^- x^- = \sigma_x |x|$, $\sigma_u^+ u^+ - \sigma_u^- u^- = \sigma_u |u|$ and one retrieves the symmetric CVIU model studied in [35] for the scalar case.

In the CVIU class of models, the modeling errors are associated to extra independent BM driven terms that are modulated by the state and control equilibrium offsets, according with (13). Here the model takes the unsymmetrical and more general form than in [35] because in the fishery management problem it is desirable that the uncertainty on the nature could be stressed asymmetrically in the model. If the biomass is above of what is the assigned equilibria for one reason or another, it is not so critical as if it falls below the equilibria, in the sense that the depletion due to harvesting may be getting too harsh on the population number. No matter if its due to a coarse model parametrization or other effects that are not accounted for, the uncertainty here is not symmetrically taken into the model, by setting $\sigma_x^- > \sigma_x^+$ in the model in (13). Similarly, the uncertainty should be amplified if the decision is to harvest more than the agreed rate h_e, compared to the decision of harvesting less than what had been previously consented, and $\sigma_u^- < \sigma_u^+$ in the model.

The management problem for fisheries is seen as the choice of a harvesting nominal point h_e, a rule that can be reviewed to a value $h(t) = u(t) + h_e$ at any time $t \geq 0$, subject to the unknown dynamics in (11), which is "roughly modeled" by (13). Even for the simplest logistic model, possibly $r \neq r_0$ and $K \neq K_0$, and the model is equipped for this realistic framework.

The robust approach is a possible way to deal with this scenario, e.g. see [26]. However since it provides policies that need to be adjusted to the worst case scenario, it is bound to be conservative from its inception. In a situation when the probability of the worst case scenario to happen is quite low, the robust policies do not account for this fact. Moreover, the policies that optimize a CVIU model are richer than the ones obtained by the robust approach based on LMIs. For a deeper discussion on that matter, see [36].

3.2 Asymmetric Scalar CVIU

With these elements, the control problem can be framed under the weak formulation of Itô diffusion stochastic control problems, cf., [37, Ch 2, sec 4]. The objective is to solve the family of problems $C_{s,x}, 0 \leq s < T, x \in O$, which amounts to determine J^*, the value function,

$$J^*(s,x) := \inf_{u(\cdot)\in\mathcal{U}[s,T]} J(s,x,u(\cdot)), \tag{15a}$$

associated to the scalar assymetric CVIU diffusion model,

$$dx(t) = (A(t)x(t) + B(t)u(t))\,dt + \hat{\sigma}(t,x(t),u(t))\,d\hat{W}(t), \tag{15b}$$

$t \in [s,T], x(s) = x \in O$, with

$$\hat{\sigma}(t,x,u) = \begin{bmatrix}\sigma(t) & \bar{\sigma}_x(t) + \sigma_x^+(t)x^+ - \sigma_x^-(t)x^- & \bar{\sigma}_u(t) + \sigma_u^+(t)u^+ - \sigma_u^-(t)u^-\end{bmatrix} \tag{15c}$$

and

$$d\hat{W}(t) = \begin{bmatrix}dW(t) & dW^x(t) & dW^u(t)\end{bmatrix}^\mathsf{T},$$

with the assumption that

$$t \to \sigma_x^+(t), \sigma_x^-(t), \bar{\sigma}_x(t) \geq 0, \text{ and } t \to \sigma_u^+(t), \sigma_u^-(t), \bar{\sigma}_u(t) \geq 0, \quad t \in [0,T]. \tag{15d}$$

The goal is to find an admissible pair $u^*(\cdot) \in \mathcal{U}[s,T]$ and the corresponding $x^*(\cdot)$, such that $(x^*(\cdot), u^*(\cdot))$ achieves the minimum of $J(s,x,u(\cdot))$ over $\mathcal{U}[s,T]$.

The elements above place the control problems $C_{s,x}$ for a more general asymmetric CVIU than we wish to apply to the fishery harvesting model. The idea here is to explore, within the general framework, important features of the control problem, and whenever possible, we quote from the results in [35].

3.3 HJB equation and General Solution

The elements above place the control problems $C_{s,x}$ for the asymmetric CVIU model, and the first important aspect is stated in the next proposition. The proof of it in the case of symmetric multivariable CVIU model is presented in [35], and the proof for the asymmetric case is derived without much effort, thus omitted.

Proposition 1. *Under the choice in* (15d) *and the assumption that the running cost, $f(t,\cdot,\cdot), t \in [0,T]$, and the final cost, $g(\cdot)$, are (strictly) convex functions, it follows that the cost $J(\cdot,\cdot,\cdot)$ and the value $J^*(\cdot,\cdot)$ of the problems $C_{s,x}$ are continuous and $J(t,\cdot,\cdot), J^*(t,\cdot)$ are (strictly) convex for each $t \in [0,T]$.*

Next, the dynamic programming approach associates the value of problems $C_{s,x}$ to the second order Hamilton-Jacobi-Bellman (HJB) equation. For each $(t,x,u,p,P) \in [0,T] \times \mathbb{R}^4$ define the Hamiltonian function as,

$$H(t,x,u,p,P) := \frac{1}{2} P \hat{\sigma}(t,x,u)^2 + p(A(t)x + B(t)u) + f(t,x,u) \tag{16}$$

and provided that $J^* \in C^{1,2}([0,T] \times \mathbb{R})$, one can prove that J^* is the unique solution of

$$-\frac{\partial \varphi}{\partial t} - \inf_{u \in U} H(t,x,u,\varphi_x,\varphi_{xx}) = 0, \quad \varphi(T,x) = g(x), \tag{17}$$

where φ_x and φ_{xx} indicate respectively, the gradient and the Hessian matrix of φ at (t,x). In the realm of continuous solution to the differential equation in (17) not necessarily smooth, there is the notion of viscosity solutions, e.g., see [37, 38].

The CVIU model (15b) satisfies the conditions (S1') and (S2') in [37, p 177], moreover, in view of Theorem 1, it follows directly from [38, Cor 8.1 p.221] and [37, Th 6.2 p.198] that (17) can be coined in terms of a viscosity HJB equation and J^* is the viscosity solution. In particular, from the fact that the value function J^* is convex on $O \subset \mathbb{R}$ and from the definition of the super- and subdifferentials, $(q,p,P) \in D^{1,2,\mathrm{sp}}_{t^+,x} J^*(t,x)$ is such that $P \geq 0$ or $D^{1,2,\mathrm{sp}}_{t^+,x} J^*(t,x)$ is empty. On the other hand, $D^{1,2,\mathrm{sb}}_{t^+,x} J^*(t,x)$ is never empty. As result, the differentials in (17) can be replaced by parabolic super- and subdifferentials in corresponding HJB inequalities, and hereafter the HJB in (17) is thus referred.

4 The Design of a CVIU Controller

Before going, we define some auxiliary elements, notations and equivalences that will be necessary to the next sections.

Set for $v \in \mathbb{R}$, let $\mathcal{S}^+(v) : \mathbb{R} \to \{0, +1\}$ and $\mathcal{S}^-(v) : \mathbb{R} \to \{0, -1\}$ the sign functions defined as

$$\mathcal{S}^+(v) = \begin{cases} +1, & \text{if } v > 0 \\ 0, & \text{otherwise} \end{cases} \qquad \mathcal{S}^-(v) = \begin{cases} -1, & \text{if } v < 0 \\ 0, & \text{otherwise} \end{cases} \tag{18}$$

respectively. We also employ the notation $\mathcal{S}(v)$ to express $\mathcal{S}^+(v) + \mathcal{S}^-(v)$, and also add the subscript $\mathcal{S}_x$ and $\mathcal{S}_u$ whenever necessary.

For the use of representing subdifferentials, we also consider similar definitions $\tilde{\mathcal{S}}^+(v)$ and $\tilde{\mathcal{S}}^-(v)$ that are sets defined as

$$\tilde{\mathcal{S}}^+(v) := \begin{cases} +1, & \text{if } v > 0, \\ [0, +1], & \text{otherwise.} \end{cases} \qquad \tilde{\mathcal{S}}^-(v) := \begin{cases} -1, & \text{if } v < 0, \\ [-1, 0], & \text{otherwise.} \end{cases} \tag{19}$$

respectively.

Proposition 2. *For $A \in \mathbb{R}$ and $v \in \mathbb{R}$, the following subdifferentials with respect to v can be calculated:*

$$D_v^{1,\mathrm{sb}}\left(A(v^+)\right) = A\tilde{\mathcal{S}}^+(v) \tag{20}$$

$$D_v^{1,\mathrm{sb}}\left(A(v^-)\right) = -A\tilde{\mathcal{S}}^-(v) \tag{21}$$

and

$$D_v^{1,\mathrm{sb}}\left(A(v^+)^2\right) = 2Av^+ \tag{22}$$

$$D_v^{1,\mathrm{sb}}\left(A(v^-)^2\right) = 2Av^- \tag{23}$$

We inspect here the pointwise minimization in the HJB equation (17) that involves the Hamiltonian function H in (16). We write (16) equivalently as

$$H(t, x, u, p, P) = f(t, x, u) + p(A(t)x + B(t)u) \\ + \frac{1}{2}\Big(\mathit{\Gamma}_0(t, P) + \mathit{\Gamma}_1(t, x, u, P) + \mathit{\Gamma}_2(t, x, u, P)\Big) = 0, \tag{24a}$$

with the notation,

$$\begin{aligned}
\Gamma_0(t,P) :=& P\left(\sigma(t)^2+\bar{\sigma}_x(t)^2+\bar{\sigma}_u(t)^2\right), && \text{(24b)}\\
\Gamma_1(t,x,u,P) :=& 2P\left(\bar{\sigma}_x(t)\left(\sigma_x^+(t)x^+-\sigma_x^-(t)x^-\right)\right.\\
&\left.+\bar{\sigma}_u(t)\left(\sigma_u^+(t)u^+-\sigma_u^-(t)u^-\right)\right), && \text{(24c)}\\
\Gamma_2(t,x,u,P) :=& P\left(\left(\sigma_x^+(t)x^+\right)^2+\left(\sigma_x^-(t)x^-\right)^2\right.\\
&\left.+\left(\sigma_u^+(t)u^+\right)^2+\left(\sigma_u^-(t)u^-\right)^2\right) && \text{(24d)}
\end{aligned}$$

Note that the values u^+ and u^- appearing in (24) makes the Hamiltonian not differentiable w.r.t. u at the origin. The notion of subdiffferentials provides the tools to deal with the minimization w.r.t u in the HJB equation.

Let us consider the HJB equation in (17) for the Hamiltonian function as in (24), applied to a test function $v(\cdot,\cdot)$, which is continuous and convex on the second argument.

Lemma 1. *Consider $v:[0,T]\times\mathbb{R}\to\mathbb{R}$, a convex function on the second argument and $q\in\mathbb{R}, p\in\mathbb{R}$ and $P\in\mathbb{R}$, with $(q,p,P)\in D^{1,2,\mathrm{sp}}_{t+,x}v(t,x)$, for some fixed t. If such a v is the solution of the HJB equation in (17), the minimum $u=u^*$ of the HJB satisfies*

$$\begin{aligned}
0\in D_u^{1,\mathrm{sb}}H(t,x,u,p,P)|_{u=u^*} =& \frac{\partial_u f(t,x,u)}{\partial u}|_{u=u^*}+B(t)p\\
&+P\left(\bar{\sigma}_u(t)\sigma_u^+(t)\right)\tilde{\mathcal{S}}^+(u)|_{u=u^*}+P\left(\bar{\sigma}_u(t)\sigma_u^-(t)\right)\tilde{\mathcal{S}}^-(u)|_{u=u^*}\\
&+P\left(\sigma_u^+(t)^2\right)u^+|_{u=u^*}+P\left(\sigma_u^-(t)^2\right)u^-|_{u=u^*}. \qquad (25)
\end{aligned}$$

Proof. First order condition for the optimality of u in the HJB equation in (17) provides (25) (see Proposition 2). Since $P\geq 0$, the first-order condition above provides the global minimum of H on the set U, namely, the function $H(t,x,\cdot,p,P)$ is indeed convex and the minimum is attained for u^* that satisfies (25). □

Regarding (19) and (25), note that it is necessary to determine the sets $\mathcal{S}^+(u^*)$ and $\mathcal{S}^-(u^*)$ seeking to determine u^* itself. In fact, it is possible for each $t\in[0,T]$, to divide the state space line $\mathbb{R}$ into segments for which the optimal control u^* has known sign. The next result is drawn mutatis mutandis from [35], and thus, the proof is omitted.

Proposition 3. *Consider that $f(t,\cdot,\cdot), t\in[0,T]$ and $g(\cdot)$ are convex and $f(t,x,\cdot)$ is continuously differentiable for each t and x. Then the optimal control u^* for the problem in (15) satisfies the* *following sign conditions,*

$$\begin{cases} u^* > 0, \textit{if } x \in \mathcal{R}^+(t), \textit{ with} \\ \mathcal{R}^+(t) := \big\{ x \in \mathbb{R} : \lim_{u \downarrow 0} \partial H(t,x,u,p,P)/\partial u < 0 \big\}, \\ u^* < 0, \textit{if } x \in \mathcal{R}^-(t), \textit{ with} \\ \mathcal{R}^-(t) := \big\{ x \in \mathbb{R} : \lim_{u \uparrow 0} \partial H(t,x,u,p,P)/\partial u > 0 \big\}, \\ u^* = 0, \textit{if } \ x \in \mathcal{R}^0(t) := \big\{ x \in \mathbb{R} : (\mathcal{R}^+(t) \cup \mathcal{R}^-(t))^c \big\}, \end{cases} \tag{26}$$

for some p and P such that $(q,p,P) \in D^{1,2,\mathrm{sp}}_{t+,x} J^*(t,x)$, a.e. $t \in [0,T], \mathbf{P}$-a.s.

4.1 The Inaction Region

Note from Proposition 3 that each region $\mathcal{R}^0(t), i = 1, \ldots, r$ can be equivalently represented by

$$\mathcal{R}^0(t) = \Big\{ x \in \mathbb{R} : \lim_{u \uparrow 0} \frac{\partial H(t,x,u,p,P)}{\partial u} \leq 0 \leq \lim_{u \downarrow 0} \frac{\partial H(t,x,u,p,P)}{\partial u} \Big\}. \tag{27}$$

Note also that $u^*(t,x) \equiv 0, \forall x \in \mathcal{R}^0(t)$ holds and for this reason, $\mathcal{R}^0(\cdot)$ (or simply $\mathcal{R}^0$) is called *Inaction Region* for the control input. It is important to emphasize that "inaction" does mean that a *zero-control variation* should be optimal inside region $\mathcal{R}^0$, namely, $u^*(t) = 0 \Rightarrow h(t) = h_e$.

It could happen though, that there would be no gap on the inequalities in (27), which leads to $\mathcal{R}^0$ to be no more than a point in the state line. The next lemma gives conditions for the inaction region $\mathcal{R}^0(t)$ indeed exists as an interval with nonzero length at the line, from the fact that $J^*(t,\cdot)$ is a convex function.

Lemma 2. *Consider* (15d) *and that $f(t,x,\cdot)$ is a continuously differentiable function for each t and x. Let (p,P) with $(q,p,P) \in D^{1,2\,\mathrm{sp}}_{t+,x} J^*(t,x)$, for $(t,x) \in [0,T] \times \mathbb{R}$. If*

$$\begin{aligned} &\textit{either,} \quad && \delta^+(t) := P\bar{\sigma}_u(t)\sigma_u^+(t) > 0, \\ &\textit{or,} && \delta^-(t) := P\bar{\sigma}_u(t)\sigma_u^-(t) > 0, \end{aligned} \tag{28}$$

the region $\mathcal{R}^0(t) \subset \mathbb{R}$ is an interval defined by

$$\mathcal{R}^0(t) = \Big\{ x \in \mathbb{R} : -\delta^+(t) \leq \frac{\partial f(t,x,u)}{\partial u}\Big|_{u^*=0} + B(t)p \leq \delta^-(t) \Big\}. \tag{29}$$

Note that $P\bar{\sigma}_u(t)\sigma_u^+(t) \geq 0$ and $P\bar{\sigma}_u(t)\sigma_u^-(t) \geq 0$ hold as a consequence of (15d) and the fact that $P \geq 0$. It follows from the lemma that if $P\bar{\sigma}_u\sigma_u^+ > 0$ or $P\bar{\sigma}_u\sigma_u^- > 0$, then there exists a region $\mathcal{R}^0(t) \subset \mathbb{R}$ with nonzero length in the state space line.

Proof. Consider the definition of the regions $\mathcal{R}^0(t)$ in (27). By applying the limits to the Hamiltonian function derivatives in (25), (see (24)), we obtain

$$\lim_{u\uparrow 0} \frac{\partial H(t,x,u,p,P)}{\partial u} = \lim_{u\uparrow 0}\left[\frac{\partial f(t,x,u)}{\partial u} + P\sigma_u^-(t)^2 u\right] + B(t)p - P\bar{\sigma}_u(t)\sigma_u^-(t) \leq 0, \tag{30}$$

$$\lim_{u\downarrow 0} \frac{\partial H(t,x,u,p,P)}{\partial u} = \lim_{u\downarrow 0}\left[\frac{\partial f(t,x,u)}{\partial u} + P\sigma_u^+(t)^2 u\right] + B(t)p + P\left(\bar{\sigma}_u(t)\sigma_u^+(t)\right) \geq 0. \tag{31}$$

Thus, the condition for the inaction gap to appear is that at least $\delta^+(t)$ or $\delta^-(t)$ be positive. $\square$

The corollary in the sequel summarizes the results of the section.

Corollary 1. *The optimal control policy for the CVIU problem in* (15) *satisfies* (25) *for some* $q \in \mathbb{R}, p \in \mathbb{R}$ *and* $P \in \mathbb{R}_+$, *with* $(q,p,P) \in D^{1,2,\mathrm{sp}}_{t+,x}J^*(t,x)$. *The sign of* u^* *at* (t,x) *is determined by* (26), *and* (29) *defines the inaction region* ($u^* = 0$) *for the input control.*

The corollary outlines the general behavior of the optimal solution for problem (15) and next, we proceed on advances for the control synthesis for the time-invariant infinite time horizon problem.

4.2 Discounted Quadratic-linear Running Cost

Let us consider the control problem $C_{s,x}$ in (15) and the previous results in an infinite time horizon framework. In this setting, A, B in the drift coefficient, and $\sigma, \sigma_x, \bar{\sigma}_x, \sigma_u$ and $\bar{\sigma}_u$ in the diffusion coefficient are assumed to be time-invariant. The performance of the system is measured through the expected discounted cost, with running quadratic and linear costs,

$$J(s,x,u(\cdot)) = E^x\left[\int_s^\infty e^{-\alpha t}\left(Qx(t)^2 + qx(t) + Ru(t)^2 + ru(t)\right)dt\right], \tag{32a}$$

for some $Q \geq 0, R > 0, \alpha > 0$ and arbitrary q and r, associated to the time-invariant CVIU model,

$$dx(t) = (Ax(t) + Bu(t))\,dt + \hat{\sigma}(x(t),u(t))d\hat{W}(t), \tag{32b}$$

$t \in [s,\infty), x(s) = x$ nonrandom, and $\hat{\sigma}(x,u)$ as in (15c) comprising time-invariant parameters. Suppose that $J(0,x,u(\cdot)) < \infty$ for some $u(\cdot) \in \mathcal{U}[s,\infty)$, and the final goal is to find an admissible pair

$u^*(\cdot) \in \mathcal{U}[s,\infty)$ and the corresponding $x^*(\cdot)$ such that the minimum of $J(s,x,u(\cdot))$ is achieved over the class $\mathcal{U}[s,\infty)$. The value function of this problem is defined analogously to the finite horizon case.

Let $\{O_m : m \in \mathbb{Z}_+\}$ be a fixed family of open precompact[1] sets for which $O_m \uparrow \mathbb{R}$ as $m \to \infty$. Let T_m denote the first entrance time to O_m^c (maybe $+\infty$) and ζ is the exit time for the process,

$$\zeta := \lim_{m\to\infty} T_m$$

The process $x(\cdot)$ is *non-explosive* if $P_x(\zeta = \infty) = 1, \forall x \in \mathbb{R}$, and one can in principle adapt the infinite horizon problem to a T_m-stopped problem. Let,

$$J_{T_m}(s,x,u(\cdot)) = E^x\Big[\int_s^{T_m} e^{-\alpha t}\left(Qx(t)^2 + qx(t) + Ru(t)^2 + ru(t)\right)dt + e^{-\alpha T_m} g(x(T_m))\Big].$$

One seeks a time-invariant solution $V^*(x)$ related to $J^*(t,x)$ through an exponential factor, such that one sets $g = V^*$ above, and for $x(s) = x$ and for each $s < T_m$,

$$J^*(s,x) = \lim_{m\to\infty} \inf_{u(\cdot)\in\mathcal{U}[s,\infty)} J_{T_m}(s,x,u(\cdot)) = e^{-\alpha s}V^*(x). \tag{33}$$

This "coupling" is justified, provided that the process is strong Markov and that $\limsup_{m\to\infty} E^x[e^{-\alpha T_m}V^*(x(T_m))] = 0$ holds. This is dealt with the notions of non-explosiveness and stable solutions, see [35, sec IV-C] for more details.

The announced control problem is denoted by $C_{\infty,x}$. If $V^* \in C^{1,2}(\mathbb{R})$, the HJB equation in (17) is written as

$$\alpha\varphi - \inf_{u\in U} H(x,u,\varphi_x,\varphi_{xx}) = 0 \tag{34}$$

and V^* would be the unique solution of (34). The Hamiltonian function here is as in (24) with $f(t,x,u) = Qx^2 + qx + Ru^2 + ru$ and t-independent functions $\Gamma_0(P), \Gamma_1(x,u,P)$ and $\Gamma_2(x,u,P)$. As in Section 3.3 one cannot presume that $V^* \in C^{1,2}(\mathbb{R})$ and the solution for the HJB equation (34) should be given in terms of viscosity solution. Theorem 1 applied here assures that V^* is convex function.

We use the following definitions in the sequel. Let $\mathcal{H} : \mathbb{R}^2 \times \{-1,0,+1\} \to \mathbb{R}$, $\mathcal{G} : \mathbb{R}^2 \times \{-1,0,+1\} \to$

[1] O is a precompact set if the closure of O is compact.

$\mathbb{R}$, $\mathcal{Q} : \mathbb{R}^2 \times \{-1, 0, +1\} \to \mathbb{R}$ and $\mathcal{R} : \mathbb{R}^2 \times \{-1, 0, +1\} \to \mathbb{R}$ be the following operators

$$\mathcal{H}(t, U, s) = 2U\bar{\sigma}_x(t) \left(\sigma_x^+(t)s^+ + \sigma_x^-(t)s^-\right) \tag{35}$$

$$\mathcal{G}(t, U, s) = 2U\bar{\sigma}_u(t) \left(\sigma_u^+(t)s^+ + \sigma_u^-(t)s^-\right) \tag{36}$$

$$\mathcal{Q}(t, U, s) = U \left(\left(\sigma_x^+(t)s^+\right)^2 + \left(\sigma_x^-(t)s^-\right)^2\right) \tag{37}$$

$$\mathcal{R}(t, U, s) = U \left(\left(\sigma_u^+(t)s^+\right)^2 + \left(\sigma_u^-(t)s^-\right)^2\right) \tag{38}$$

Note that if $U \geq 0$ then $\mathcal{Q}(\cdot, U, \cdot), \mathcal{R}(\cdot, U, \cdot) \geq 0$.

Proposition 4. *$\Gamma_1(t, x, u, P) = \mathcal{H}(t, P, \mathcal{S}(x))x + \mathcal{G}(t, P, \mathcal{S}(u))u$ and*

$$\Gamma_2(t, x, u, P) = \mathcal{Q}(t, U, \mathcal{S}(x))x^2 + \mathcal{R}(t, U, \mathcal{S}(u))u^2$$

hold. Eq. (34) *can be written equivalently as (see* (24)*),*

$$\begin{aligned}\alpha\varphi(x) - \inf_{u \in U} \Big\{ (Q + \frac{1}{2}\mathcal{Q}(P, \mathcal{S}(x)))x^2 + (R + \frac{1}{2}\mathcal{R}(P, \mathcal{S}(u)))u^2 \\ + \frac{1}{2}\mathcal{H}(P, \mathcal{S}(x))x + \frac{1}{2}\mathcal{G}(P, \mathcal{S}(u))u \\ + p(Ax + Bu) + qx + ru + \frac{1}{2}\Gamma_0(P) = 0\end{aligned} \tag{39}$$

with $p = \varphi_x$ and $P = \varphi_{xx}$ if φ is twice differentiable at x. Otherwise, for each $(p, P) \in D_x^{1,2\,\mathrm{sp}}\varphi(x)$ (39) *holds with '≤ 0' (φ is a viscosity subsolution), and for each $(p, P) \in D_x^{1,2\,\mathrm{sb}}\varphi(x)$* (39) *holds with '$\geq 0$' ($\varphi$ is a viscosity supersolution).*

Remark 1. Since $\varphi(\cdot)$ that satisfies (39) in the classic or in the viscosity sense is such that $\varphi = V^*$, it is a convex function and therefore, either $\varphi_{xx} \geq 0$, or for each P with $(p, P) \in D_x^{1,2\,\mathrm{sp}}\varphi(x)$ one has $P \geq 0$.

From Section 4.1, the optimal control of the CVIU model can be related to three distinct regions on the state space $\mathcal{R}^+, \mathcal{R}^-$ and $\mathcal{R}^0$ for the control input u, such that the sign of the optimal u^* is positive, negative and zero respectively. In the present setting these regions are time-invariant and the idea is to check if the value V^* coincides piecewisely with some quadratic solutions, by checking if the HJB equation (39) satisfies this type of solution within the three distinct regions.

4.3 Characterization of the inaction region

In Section 4.1 a region of inaction $\mathcal{R}^0 \in \mathbb{R}$ for control input u was identified. Here, it is time-invariant and possibly a nonzero length interval of the real line. Recall that the time-invariant solution $V^*(\cdot)$ is convex, and one can conclude that its minimum is attained next to the origin and near region $\mathcal{R}^0$. The idea is to solve the problem (32) inside $\mathcal{R}^0$, relying on the behavior of V^* sufficiently close to an arbitrary point x_0 inside $\mathcal{R}^0$, and realize that the choice of $x_0 \to 0$ does not affect the representation. Let us consider the following type of equation,

$$(2A-\alpha)X + Q + \mathcal{Q}\,(X,s) = 0, \tag{40}$$

with $s = +1$ or $s = -1$.

Lemma 3. *Consider the problem $C_{\infty,x}$ in* (32)*–*(33)*. Suppose that*

$$X^+ = -\frac{Q}{2A-\alpha+(\sigma_x^+)^2}, \quad \textit{or}, \quad X^- = -\frac{Q}{2A-\alpha+(\sigma_x^-)^2}, \tag{41}$$

are such that $X^+, X^- > 0$. Then set

$$v_0^+ = -\frac{q+2X^+\bar{\sigma}_x\sigma_x^+}{A-\alpha} \quad \textit{or}, \quad v_0^- = -\frac{q-2X^-\bar{\sigma}_x\sigma_x^-}{A-\alpha} \tag{42}$$

The inaction region $\mathcal{R}^0$ is the connected interval,

$$\begin{aligned}\mathcal{R}_0 =& \{x \le 0 : -\bar{\sigma}_u\sigma_u^+ \le Bx + (Bv_0^- + r)/2X^- \le \bar{\sigma}_u\sigma_u^-\} \\ &\cup \{x \ge 0 : -\bar{\sigma}_u\sigma_u^+ \le Bx + (Bv_0^+ + r)/2X^+ \le \bar{\sigma}_u\sigma_u^-\}.\end{aligned} \tag{43}$$

If X^+ or X^- is not positive, the corresponding subinterval is empty.

Proof. Let us assume that point $x \in \mathcal{R}_0$ and knowing that $u^* = 0$, (39) yields,

$$\alpha V^*(x) - \Big\{(Q + \frac{1}{2}\mathcal{Q}(P, \mathrm{sign}(x))x^2 + \frac{1}{2}\mathcal{H}(P, \mathrm{sign}(x))x + (pA+q)x + \frac{1}{2}\Gamma_0(P)\Big\} = 0 \tag{44}$$

with $p = V_x^*$ and $P = V_{xx}^*$ if V^* is twice differentiable at x; otherwise, $(p', P') \in D_x^{1,2\,\mathrm{sp}}V^*(x)$ and $(p'', P'') \in D_x^{1,2\,\mathrm{sb}}V^*(x)$ with (44) changed to corresponding inequalities.

It was already hinted that the time-invariant value function $V^*(\cdot)$ inherits convexity as result of Theorem 1, and a quadratic function provides a good approximation to the optimal cost in a neighborhood of any point x_0. Set here $x_0 \leq 0$ or $x_0 \geq 0$ and a sequence $\{\zeta_k\}_{k\geq 0}$ with each ζ_k an Alexandrov point of function V^* and $\zeta_k \to x_0$ having homogenous signals, in such a way that $\text{sign}(\zeta_k) = -1, \forall k$ for the former, or $\text{sign}(\zeta_k) = +1, \forall k$, for the latter choice. Let us set a quadratic approximation for V^* near x_0 and for some $\epsilon > 0$ and k sufficiently large, we write,

$$\left| V^*(x) - \left(V^*(\zeta_k) + V_x^*(\zeta_k)(x - x_0) + \frac{1}{2} V_{xx}^*(\zeta_k)(x - x_0)^2 \right) \right| < \epsilon,$$

provided that $x \in \mathcal{R}_0$, and $\|x - x_0\| < \delta = \delta(\epsilon)$. It follows from Alexandrov theorem for convex functions [39, Th 3.11.2] that $0 \leq V_{xx}^*(\zeta_k) \leq \rho I$, for some $\rho > 0$.

Consider a point x in the neighborhood of x_0 as above. Substituting the approximation of V^* at such a x near x_0 into (44), one can state that

$$\begin{aligned} &\alpha \left(V^*(\zeta_k) + V_x^*(\zeta_k)(x - x_0) + \frac{1}{2} V_{xx}^*(\zeta_k)(x - x_0)^2 \right) \\ &\qquad - \Big\{ (Q + \frac{1}{2} \mathcal{Q}(V_{xx}^*(\zeta_k), \text{sign}(\zeta_k)) x^2 + \frac{1}{2} \mathcal{H}(V_{xx}^*(\zeta_k), \text{sign}(\zeta_k)) x + qx \\ &\qquad\qquad + (V_{xx}^*(\zeta_k)(x - x_0) + V_x^*(\zeta_k)) A x + \frac{1}{2} \varGamma_0(V_{xx}^*(\zeta_k)) \Big\} \end{aligned} \tag{45}$$

is dominated from above and from below by some $\pm\gamma(\epsilon)$ with $\gamma(\epsilon) \to 0$ as $\epsilon \to 0$. Rearranging (45),

$$\begin{aligned} &\frac{1}{2} \Big((2A - \alpha) V_{xx}^*(\zeta_k) + 2Q + \mathcal{Q}(V_{xx}^*(\zeta_k), \text{sign}(\zeta_k)) \Big) x^2 \\ &\qquad + \Big(q + \frac{1}{2} \mathcal{H}(V_{xx}^*(\zeta_k), \text{sign}(\zeta_k)) + (\alpha - A) V_{xx}^*(\zeta_k) x_0 + (A - \alpha) V_x^*(\zeta_k) \Big) x \\ &\qquad\qquad - \alpha V^*(\zeta_k) + \frac{1}{2} \varGamma_0(V_{xx}^*(\zeta_k)) + \alpha V_x^*(\zeta_k) x_0 - \frac{1}{2} \alpha V_{xx}^*(\zeta_k) x_0^2. \end{aligned} \tag{46}$$

Taking the limit of $\{\zeta_k\}_{k\geq 0}$, the choices $V_{xx}^*(x) = 2X^s$ and $V_x^*(x) = v^s$, with $s = -1$ or $s = +1$ as appropriate, the above expression reads as:

$$\begin{aligned} &\Big((2A - \alpha) X^s + Q + \mathcal{Q}(X^s, s) \Big) x^2 \\ &\qquad + \Big(q + \mathcal{H}(X^s, s) + (\alpha - A)(2X^s x_0 - v^s) \Big) x \\ &\qquad\qquad - \alpha V^*(x_0) + \varGamma_0(X^s) - \alpha (X^s x_0^2 - v^s x_0). \end{aligned} \tag{47}$$

Suppose that $(2A - \alpha) X^s + Q + \mathcal{Q}(X^s, s) = 0$ has a positive solution for X^s. Then one can set the Hamiltonian to zero by a qualified choice of the parameter v^s and obtain the corresponding value of

$V^*(x_0)$ from (47). That provides the quadratic representation for the value function,

$$V^*(x) = X^s(x-x_0)^2 + v^s(x-x_0) + \frac{\Gamma_0(X^s)}{\alpha} - (X^s x_0^2 + v^s x_0) \tag{48}$$

valid for each $x \in \mathcal{R}_0$ in a neighborhood of x_0 of same signal. Note that one may choose x_0 such that $\text{sign}(x_0) = s$, but otherwise arbitrarily, to set (46) to zero.

Recall the definition of region $\mathcal{R}^0$ in (29), Lemma 2. In the present situation, with quadratic running costs, the HJB equation in (39) specialized for region $\mathcal{R}^0$ in (44) for the time-invariant solution, V^* in (48), one obtains $P = 2X^s$, $p = 2X^s(x-x_0)+v^s$ with $v^s = -\frac{q+\mathcal{H}(X^s,s)}{A-\alpha} + 2X^s x_0$, valid for $x, x_0 \in \mathcal{R}^0$ of same signals. Now let $x_0 \uparrow 0$ if $s=-1$, and $x_0 \downarrow 0$ if $s=+1$, to obtain form Lemma 2 the characterization,

$$\begin{aligned} \mathcal{R}_0 = \{x \le 0 : -2X^-\bar{\sigma}_u\sigma_u^+ \le B(2X^-x+v_0^-)+r \le 2X^-\bar{\sigma}_u\sigma_u^-\}\cup \\ \{x \ge 0 : -2X^+\bar{\sigma}_u\sigma_u^+ \le B(2X^+x+v_0^+)+r \le 2X^+\bar{\sigma}_u\sigma_u^-\} \end{aligned} \tag{49}$$

with $v_0^+ = -\frac{q+\mathcal{H}(X^+,+)}{A-\alpha}$ and $v_0^- = -\frac{q+\mathcal{H}(X^-,-)}{A-\alpha}$ as in (42). Rearranging (49), we obtain (43).

It remains to show that $\mathcal{R}^0$ expressed by (49) is a connected region. Note for this purpose that $x \to V_x^*(x)$ is defined for $x<0$ or $x>0$ inside $\mathcal{R}^0$, and it should be monotone increasing function from the fact that V^* is convex (piecewise quadratic) function[2]. If $B>0$, consider the upper limit of the first interval and the lower limit of the second one. Otherwise, consider the lower limit of the first interval and the upper limit of the second one. Region $\mathcal{R}_0$ will be connect provided that

$$B>0: \begin{cases} BV_x^-(z-\frac{\bar{\sigma}_u\sigma_u^-}{B})+r \le 0, \\ BV_x^+(y+\frac{\bar{\sigma}_u\sigma_u^+}{B})+r \ge 0. \end{cases} \qquad B<0: \begin{cases} BV_x^-(z+\frac{\bar{\sigma}_u\sigma_u^+}{B})+r \ge 0, \\ BV_x^+(y-\frac{\bar{\sigma}_u\sigma_u^-}{B})+r \le 0. \end{cases}$$

with $z \ge y$.

Let us assume that the interval $\mathcal{R}_0$ is not connected for the case $B>0$. This is equivalent to assume that there exist a z^+ with $BV_x^-(z^+ - \bar{\sigma}_u\sigma_u^-/B)+r=0$ and a y^- with $BV_x^+(y^- + \bar{\sigma}_u\sigma_u^+/B)+r=0$ with $z^+ < y^-$. It then follow that

$$V_x^-(z^+ - \frac{\bar{\sigma}_u\sigma_u^-}{B}) = V_x^+(y^- + \frac{\bar{\sigma}_u\sigma_u^+}{B}) = -\frac{r}{B}$$

[2]A generalized verson of the monotocity pointed out can be developed when V^* is not differentiable but convex. We choose to write these arguments when V^* is continuously differentiable and V_x^* exists as a standard function, to not impair the clarity with too many technical details. However, they can be shown in the general case.

Since $x \to V_x^*(x)$ is monotone increasing function, the above would imply that $V_x^*(x) = -r/B$ in the interval $z^+ - \bar{\sigma}_u \sigma_u^-/B \leq x \leq y^- + \bar{\sigma}_u \sigma_u^+/B$ and hence $V_{xx}^*(x) = 0$ in the same interval. However, from the HJB equation (39) comes the fact that the optimal control would be $u^*(x) = 0$ in the same interval, which denies the assumption and implies the fact that region $\mathcal{R}_0$ is a connected interval.

When $B < 0$ the proof follows exactly the same arguments considering the interval $z^+ + \bar{\sigma}_u \sigma_u^+/B \leq x \leq y^- - \bar{\sigma}_u \sigma_u^-/B$, and the details are omitted. □

4.4 Asymptotic solutions for regions with homogeneous control signs

In previous section the optimal solution was obtained inside region $\mathcal{R}^0$ of the state space line, a region containing the minimal cost. When it is nonempty, this region separates two distinct regions for which either $u^*(x) > 0$ or $u^*(x) < 0$, denoted respectively by $\mathcal{R}^+$ and $\mathcal{R}^-$

Here the situation in which $u^* \neq 0$ is considered, whether the interval $\mathcal{R}^0$ is of zero length or not. If one takes points $x \in \mathbb{R}$ such that $|x|$ is sufficiently large, the control sign is either $u_i^*(\cdot) > 0$ or $u_i^*(\cdot) < 0$, and it remains the same for a large neighborhood of such a point x.

In parallel to Section 4.3, we seek steady state solutions for the HJB equation inside regions $\mathcal{R}^+$ and $\mathcal{R}^-$ for $|x| > L$ with sufficiently large L. Here we consider the following equations, in $X^{s_x}, v^{s_x,u}$ and $\ell^{s_x,u}$,

$$(2A - \alpha)X^{s_x} - \frac{(BX^{s_x})^2}{R + \mathcal{R}(X^{s_x}, s_u)} + Q + \mathcal{Q}(X^{s_x}, s_x) = 0, \tag{50a}$$

$$\Big(A - \alpha - \frac{B^2 X^{s_x}}{R + \mathcal{R}(X^{s_x}, s_u)}\Big) v^{s_x,u} - \frac{BX^{s_x}}{R + \mathcal{R}(X^{s_x}, s_u)} \big(\mathcal{G}(X^{s_x}, s_u) + r\big) + \mathcal{H}(X^{s_x}, s_x) + q = 0 \tag{50b}$$

$$-\alpha \ell^{s_x,u} + \varGamma_0(X) - \frac{1}{4(R + \mathcal{R}(X, s_u))} (Bv^{s_x,u} + \mathcal{G}(X^{s_x}, s_u) + r)^2 = 0 \tag{50c}$$

for certain combinations of $s_x = \pm 1$ and $s_u = \pm 1$. Consider also the feedback control

$$u(x) = -\frac{1}{R + \mathcal{R}(X^{s_x}, s_u)} \Big(BX^{s_x} x + \frac{1}{2}(Bv^{s_x,u} + \mathcal{G}(X^{s_x}, s_u) + r)\Big) \tag{51}$$

The idea of next lemma is to present some simple asymptotic solution for the HJB equation valid when $L \to \infty$.

Lemma 4. *Consider the problem $C_{\infty,x}$ in* (32)–(33)*. If there exists a solution for* (50a) *with* $X^{s_x} > 0$, *then the value V^* tends asymptotically to $V_a^{s_x,u}$ for $|x| > L \to \infty$, where,*

$$V_a^{s_x,u}(x) := X^{s_x} x^2 + v^{s_x,u} x + \ell^{s_x,u}, \tag{52}$$

with $Xs_x, v^{s_x,u}$ and $\ell^{s_x,u}$ as in (50) *for suitable combination of signals $s_x \pm 1, s_u = \pm 1$. Moreover, for $|x| > L$ with $L \to \infty$, the optimal control policy $u^*(x)$ tends asymptotically to $u(x)$ in* (51).

Proof. We take into account the HJB equation (39) in Proposition 4. As in Section 4.3, let us employ the quadratic approximation here in a neighborhood of a point x_0 with $|x_0| > L$ for some large value of L. For this, consider the HJB equation in (39) with a quadratic approximation valid for x near x_0,

$$\begin{aligned}
&\alpha\left(V^*(\zeta_k) + V_x^*(\zeta_k)(x - x_0) + \frac{1}{2}V_{xx}^*(\zeta_k)(x - x_0)^2\right) \\
&\qquad - \inf_{u\in U}\Big\{(Q + \frac{1}{2}\mathcal{Q}(V_{xx}^*(\zeta_k), \operatorname{sign}(\zeta_k))x^2 + \frac{1}{2}\mathcal{H}(V_{xx}^*(\zeta_k), \operatorname{sign}(\zeta_k))x + qx \\
&\qquad\quad + (R + \frac{1}{2}\mathcal{R}(V_{xx}^*(\zeta_k), \operatorname{sign}(u)))u^2 + \frac{1}{2}\mathcal{G}(V_{xx}^*(\zeta_k), \operatorname{sign}(u))u + ru \\
&\qquad\qquad + (V_{xx}^*(\zeta_k)(x - x_0) + V_x^*(\zeta_k))(Ax + Bu) + \frac{1}{2}\varGamma_0(V_{xx}^*(\zeta_k))\Big\}
\end{aligned} \tag{53}$$

Let us denote,

$$\tilde{R}_k = R + \frac{1}{2}\mathcal{R}(V_{xx}^*(\zeta_k), \operatorname{sign}(u)) \tag{54a}$$

$$\tilde{Q}_k = Q + \frac{1}{2}\mathcal{Q}(V_{xx}^*(\zeta_k), \operatorname{sign}(\zeta_k)) \tag{54b}$$

$$\tilde{\varUpsilon}_k = \varUpsilon_k^1 x + \varUpsilon_k^0 \tag{54c}$$

with $\varUpsilon_k^1 = V_{xx}^*(\zeta_k)B$, and

$$\varUpsilon_k^0 = (V_x^*(\zeta_k) - x_0 V_{xx}^*(\zeta_k))B + \frac{1}{2}\mathcal{G}(V_{xx}^*(\zeta_k), \operatorname{sign}(u)) + r$$

Rearranging (53) it gives,

$$\begin{aligned}
&\frac{1}{2}\Big((2A - \alpha)V_{xx}^*(\zeta_k) + 2\tilde{Q}_k\Big)x^2 \\
&\qquad + \Big(q + \frac{1}{2}\mathcal{H}(V_{xx}^*(\zeta_k), \operatorname{sign}(\zeta_k)) + (\alpha - A)V_{xx}^*(\zeta_k)x_0 + (A - \alpha)V_x^*(\zeta_k)\Big)x \\
&\qquad\quad - \alpha V^*(\zeta_k) + \frac{1}{2}\varGamma_0(V_{xx}^*(\zeta_k)) + \alpha V_x^*(\zeta_k)x_0 - \frac{1}{2}\alpha V_{xx}^*(\zeta_k)x_0^2 \\
&\qquad\qquad + \inf_{u\in U}\big\{\tilde{R}_k u^2 + \varUpsilon^1 xu + \varUpsilon^0 u\big\}
\end{aligned} \tag{55}$$

and since

$$\tilde{R}_k u^2 + \varUpsilon^1 x u + \varUpsilon^0 u =$$
$$(u + \frac{1}{2}\tilde{R}_k^{-1}(\varUpsilon^1 x + \varUpsilon^0))\tilde{R}_k(u + \frac{1}{2}\tilde{R}_k^{-1}(\varUpsilon^1 x + \varUpsilon^0)) - \frac{1}{4}(\varUpsilon^1 x + \varUpsilon^0)\tilde{R}_k^{-1}(\varUpsilon^1 x + \varUpsilon^0) =$$
$$\tilde{R}_k(u + \frac{1}{2}\tilde{R}_k^{-1}(\varUpsilon^1 x + \varUpsilon^0))^2 - \frac{1}{4}\tilde{R}_k^{-1}(\varUpsilon^1 x + \varUpsilon^0)^2 \quad (56)$$

and $\tilde{R}_k > 0$, one has that

$$\inf_{u\in U}\{\tilde{R}_k u^2 + \varUpsilon^1 x u + \varUpsilon^0 u\} = -\frac{1}{4}\tilde{R}_k^{-1}(\varUpsilon^1 x + \varUpsilon^0)^2$$

Hence, the optimal control at x is:

$$u^* = -\frac{1}{2}\tilde{R}_k^{-1}(\varUpsilon^1 x + \varUpsilon^0)$$

and (55) can be written as:

$$\frac{1}{2}\Big((2A-\alpha)V^*_{xx}(\zeta_k) + 2\tilde{Q}_k - \frac{1}{2}\tilde{R}_k^{-1}(\varUpsilon^1_k)^2\Big)x^2$$
$$+\Big(q + \frac{1}{2}\mathcal{H}(V^*_{xx}(\zeta_k), \mathrm{sign}(\zeta_k)) + (\alpha - A)V^*_{xx}(\zeta_k)x_0 + (A-\alpha)V^*_x(\zeta_k) - \frac{1}{2}\tilde{R}_k^{-1}\varUpsilon^1_k\varUpsilon^0_k\Big)x$$
$$-\alpha V^*(\zeta_k) + \frac{1}{2}\varGamma_0(V^*_{xx}(\zeta_k)) + \alpha V^*_x(\zeta_k)x_0 - \frac{1}{2}\alpha V^*_{xx}(\zeta_k)x_0^2 - \frac{1}{4}\tilde{R}_k^{-1}(\varUpsilon^0_k)^2 \quad (57)$$

Denote $\mathrm{sign}(x_0) = s_x$ and $\mathrm{sign}(u) = s_u$. Taking the limit of any sequence $\{\zeta_k\}_{k\geq 0}$ with each ζ_k an Alexandrov point, $\mathrm{sign}(\zeta_k) = s_x$ and $\zeta_k \to x_0$, together with the choice $V^*_{xx}(x) = 2X$ to get from (57) that

$$\Big((2A-\alpha)X + Q + \mathcal{Q}(X, s_x) - (R + \mathcal{R}(X, s_u))^{-1}(XB)^2\Big)x^2$$
$$+\Big(q + \mathcal{H}(X, s_x) + 2(\alpha - A)Xx_0 + (A-\alpha)V^*_x(x_0)$$
$$-(R + \mathcal{R}(X, s_u))^{-1}XB\{(V^*_x(x_0) - 2x_0X)B + \mathcal{G}(X, s_u) + r\}\Big)x$$
$$-\alpha V^*(\zeta_k) + \varGamma_0(X) + \alpha V^*_x(\zeta_k)x_0 - \alpha X x_0^2$$
$$-\frac{1}{4}(R + \mathcal{R}(X, s_u))^{-1}((V^*_x(x_0) - 2x_0X)B + \mathcal{G}(X, s_u) + r)^2 \quad (58)$$

holds. Now, assume that the quadratic equation,

$$(2A-\alpha)X+Q+\mathcal{Q}(X,s_x)-(R+\mathcal{R}(X,s_u))^{-1}(XB)^2=0$$

has a solution $X>0$. Since x_0 is arbitrary, except for its signal, one can set $x_0\uparrow 0$ or $\downarrow 0$ accordingly. For that, the quadratics of form $V(x)=Xx^2+vx+\ell$ with an appropriate choice of v and ℓ can be set $V^*(x)=V(x)$ and $V_x^*(x)=V_x(x)=2Xx+v$ in such a way that (58) is set to zero by simply choosing v and ℓ such that,

$$q+\mathcal{H}(X,s_x)+(A-\alpha)v-(R+\mathcal{R}(X,s_u))^{-1}XB\{Bv+\mathcal{G}(X,s_u)+r\}=0 \tag{59}$$

$$-\alpha\ell+\Gamma_0(X)-\frac{1}{4}(R+\mathcal{R}(X,s_u))^{-1}(Bv+\mathcal{G}(X,s_u)+r)^2=0 \tag{60}$$

Such a quadratic functions V defines an upper bound for the optimal V^* at the positive and negative line, since they satisfy the HJB equation at a point x with $|x|\to\infty$. $\square$

Remark 2. To set the right combination of signals of x and the optimal u one needs a matched signals reasoning. For the fishery management problem $B<0$ thus, if $x\ll 0$ then $u^*<0$ and if $x\gg 0$ then $u^*>0$ and in this case the signals are coupled, i.e., $s_x=s_u$. Writing (50) more explicitly for this case,

$$s_x=s_u=+1\qquad \begin{cases}(2A-\alpha+(\sigma_x^+)^2)X^++Q-\dfrac{(X^+B)^2}{R+X^+(\sigma_u^+)^2}=0\\ \text{Then } X^+=(-b+\sqrt{b^2-4ac})/2a, \text{ with,}\\ \quad a=(\sigma_u^+)^2(2A-\alpha+(\sigma_x^+)^2)-B^2\\ \quad b=R(2A-\alpha+(\sigma_x^+)^2)+Q(\sigma_u^+)^2,\quad c=RQ\end{cases}$$

$$\begin{cases}v^+= & -\left(A-\alpha-\dfrac{B^2X^+}{R+X^+(\sigma_u^+)^2}\right)^{-1}\Big(q+2X^+\bar{\sigma}_x\sigma_x^+(+1)\\ & \qquad -\dfrac{BX^+(r+2X^+\bar{\sigma}_u\sigma_u^+(+1))}{R+X^+(\sigma_u^+)^2}\Big)\\ \ell^+= & \dfrac{1}{\alpha}\left(\Gamma_0(X^+)-\dfrac{(Bv^++2X^+\bar{\sigma}_u\sigma_u^+(+1)+r)^2}{4(R+X^+(\sigma_u^+)^2)}\right)\end{cases}$$

$$\begin{cases}\text{Asymptotic optimal solution:}\\ u^*(x)=-\dfrac{1}{2(R+X^+(\sigma_u^+)^2)}\big(2X^+Bx+v^+B+X^+\bar{\sigma}_u\sigma_u^+(+1)+r\big)\\ \quad\text{valid when } x\to+\infty.\end{cases}$$

For $s_x = s_u = -1$ replace '+1' above by '−1' and superscripts '+' by '−'. The first equation is a modified version of the Riccati equation that appears in the discounted quadratic cost problem for deterministic controls and it resembles the ones that apply to stochastic optimal control with multiplicative BM noise. In classic problems $\sigma_x^+ = \sigma_x^- = 0$ and $\sigma_u^+ = \sigma_u^- = 0$, and the Riccati theory reveals in the scalar case that if $B \neq 0$, there exist unique solutions $X^+ > 0$ and $X^- > 0$.

Remark 3. So far the optimal solution was established inside the inaction region, and such a region was determined in Lemma 3. In addition, Lemma 4 outlines the asymptotic solution for any $|x|$ sufficiently large. The exact solution outside the region of inaction but for small values of state x is hard to find and it has to rely on the HJB equation in Proposition 4. However a good approximation is obtained at each half line by a linear interpolation between the optimal control $u^* = 0$ at the inaction region and the asymptotic control (51) obtained in Lemma 4. For more details see [35, sec. V].

Finally, we also mention that a study of the stochastic stability for solutions of the infinite horizon problem is presented in [35].

4.5 About the Inaction Region

Remark 4. The characterization in Lemma 3 above allows the following considerations:

(i) From (63) note that it is necessary that $2A - \alpha + (\sigma_x^+)^2 < 0$ to exist an inaction region for $x > 0$ and correspondently, $2A - \alpha + (\sigma_x^-)^2 < 0$ to exist an inaction region for $x < 0$. Hence, by increasing α or decreasing the uncertainty represented by σ_x^+ and σ_x^- respectively, one eventually gives rise to the inaction region in the solution.

(ii) From (43) one can understand how the length of the inaction interval varies. For the positive line write equivalently the expression in (43).

$$-\bar{\sigma}_u \sigma_u^+ \leq Bx + p_0 \leq \bar{\sigma}_u \sigma_u^-, \quad p_0 = \frac{Bv_0^+ + r}{2X^+}$$

For the case depicted in Fig. 1, $B > 0, \sigma_u^+ > \sigma_u^-$ and $e^- > 0$. In this situation, $\{x \leq 0 : -\bar{\sigma}_u \sigma_u^+ \leq Bx + (Bv_0^- + r)/2X^- \leq \bar{\sigma}_u \sigma_u^-\} = \emptyset$.

Conversely, in the negative line, if $B > 0$, and the upper end of the interval $e^+ = (\bar{\sigma}_u \sigma_u^- - p_1)/B$ with $p_1 = (Bv_0^- + r)/2X^-$ is such that $e^+ < 0$ then the set $\{x \geq 0 : -\bar{\sigma}_u \sigma_u^+ \leq Bx + (Bv_0^+ + r)/2X^+ \leq \bar{\sigma}_u \sigma_u^-\}$ must be empty.

(iii) From (ii) one can conclude that the larger $\bar{\sigma}_u, \sigma_u^+$ and σ_u^- are, the wider the inaction region will

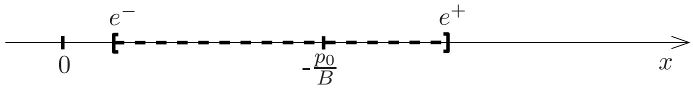

Figure 1: The inaction interval with extremes values $e^- = -(\bar{\sigma}_u \sigma_u^+ + p_0)/B$ and $e^+ = (\bar{\sigma}_u \sigma_u^- - p_0)/B$ for $B > 0$.

be; in particular, $\bar{\sigma}_u$ influences both e^- and e^+, the lower and upper ends.

The way that σ_x^+, σ_x^- and $\bar{\sigma}_x$ influence the inaction region is more complex. First recall (i) above for the role of σ_x^+ and σ_x^-. Then note that points $-p_0/B$ and $-p_1/B$ depend linearly on $\bar{\sigma}_x$, which brings them closer as $\bar{\sigma}_x$ increases.

5 Application to the bio-economic fishery model

Recall that the objective of the policymaker is to minimize

$$J(s, x, u(\cdot)) = E^x \left[\int_s^\infty e^{-\alpha t} \left(x(t)^\intercal Q x(t) + q^\intercal x(t) + u(t)^\intercal R u(t) + r^\intercal u(t) - K \right) dt \right], \tag{61a}$$

with $Q = 1$, $r = -(p - 2ch_e)$, $R = c > 0$, $K = \overline{\pi} > 0$. When $q < 0$ a reward for having a biomass surplus is introduced, otherwise, if $q > 0$ a penalty holds for a biomass increase.

About sign of r. If h_e is set to the value that maximizes the static profit, then $h_e = h_{\text{Max}} = p/2c$ and $r = 0$. If $h_e < h_{\text{Max}}$ then $r < 0$, otherwise, $h_e > h_{\text{Max}}$ and then $r > 0$. We assume that the TAC value is chosen to be smaller than the one that maximizes the static profit, thus $h_e < h_{\text{Max}}$ and $r < 0$.

The dynamic is as in (32b),

$$dx(t) = (Ax(t) + Bu(t))\, dt + \hat{\sigma}(x(t), u(t)) d\hat{W}(t), \tag{62a}$$

with

$$\hat{\sigma}(x, u) = \begin{bmatrix} \sigma & \bar{\sigma}_x + \sigma_x^+ x^+ - \sigma_x^- x^- & \bar{\sigma}_u + \sigma_u^+ u^+ - \sigma_u^- u^- \end{bmatrix} \tag{62b}$$

and

$$d\hat{W}(t) = \begin{bmatrix} dW(t) & dW^x(t) & dW^u(t) \end{bmatrix}^\intercal, \tag{62c}$$

with the assumption that $\sigma_x^+, \sigma_x^-, \bar{\sigma}_x \geq 0$, and $\sigma_u^+, \sigma_u^-, \bar{\sigma}_u \geq 0$.

Since $x(t) = z(t) - z_e$, when $x(t) > 0$, the biomass is larger than the targeted level, whereas $x(t) < 0$ means that the biomass is smaller than its desired level. It is reasonable to assume that $\sigma_x^+ \leq \sigma_x^-$, meaning that above the targeted level is somehow less "risky" than falling below. We set

$B = -1$ which means that fishing removes the biomass at the same unit rate.

Finally, it is reasonable to assume that $A \leq 0$, which means that, when there is an excess of biomass ($x(t) > 0$), any increase of the living biomass will decrease the flow of new biomass due to carrying capacity (for the logistic model it implies that $z_e \geq K/2 = z_{\mathrm{MSY}}$). It also means that when the biomass is low, $x(t) < 0$, but not too far from z_e, the biomass may grow back to its desired level. As consequence of setting $A > 0$ ($z_e < K/2$ in the logistic model), the biomass dynamics will never settle at z_e without continuous intervention, moving away from this desired level, even without fishing. In this situation, an inaction region would not be optimal.

5.1 Existence of the Inaction Region

Recall from remark *4* that the condition of existence of the Inaction Region is

$$X^+ = -\frac{Q}{2A - \alpha + (\sigma_x^+)^2} > 0, \quad \text{and,} \quad X^- = -\frac{Q}{2A - \alpha + (\sigma_x^-)^2} > 0, \tag{63}$$

Since $Q = 1 > 0$, the condition of existence is that

$$2A - \alpha + \sigma_x^{+2} < 0 \text{ or } 2A - \alpha + \sigma_x^{-2} < 0 \tag{64}$$

should be satisfied. From (64), the following propositions are straightforward.

Proposition 5 (**Biomass (State) uncertainty).** *Ceteris Paribus, when the biomass drift coefficient $A < 0$, an increase of the state uncertainty (represented by $(\sigma_x^{\pm})^2$), will reduce the chance of an inaction region to exist. In the case of $A > 0$, the dynamics implies that $x(t)$ will move away from z_e and hence inaction is no longer an optimal policy, or equivalently, the size of inaction region will be empty.*

Proposition 6 (**Discount rate).** *Ceteris Paribus, an increase of the discount rate α will increase the chance of an inaction region to exist, even if $A > 0$.*

Proposition 7 (**Stability around $\mathbf{z_e}$).** *The more (less) negative is the biomass drift coefficient A is around z_e, the more (less) likely the inaction region will exist.*

5.2 Size and position of the Inaction Region

By definition, $\bar{\sigma}_u, \sigma_u^-, \sigma_u^+, \bar{\sigma}_x, \sigma_x^+, \sigma_x^- > 0$, $B = -1 < 0$, $r < 0$, and we assume $A < 0$, meaning that z^e is stable equilibrium point. Assume also that $X^+ > 0$ and/or $X^- > 0$, i.e., the inaction region is nonempty. From Lemma 3, we know that the inaction region $\mathcal{R}^0$ is the connected interval,

$$\begin{aligned}\mathcal{R}_0 =&\mathcal{R}_0^- \cup \mathcal{R}_0^+ = \{x \leq 0 : -\bar{\sigma}_u\sigma_u^- \leq x + (v_0^- - r)/2X^- \leq \bar{\sigma}_u\sigma_u^+\} \\ &\cup \{x \geq 0 : -\bar{\sigma}_u\sigma_u^- \leq x + (v_0^+ - r)/2X^+ \leq \bar{\sigma}_u\sigma_u^+\}.\end{aligned} \tag{65}$$

with

$$v_0^+ = -\frac{q + 2X^+\bar{\sigma}_x\sigma_x^+}{A - \alpha} \quad \text{or,} \quad v_0^- = -\frac{q - 2X^-\bar{\sigma}_x\sigma_x^-}{A - \alpha} \tag{66}$$

One can easily check that the signs of v_0^+ and v_0^- will depend on the sign of q. If $q < 0$, which means a reward for the state being above the desired state level, then $v_0^- < 0$ and $v_0^+ \lesseqqgtr 0$. If $q > 0$, which means a penalty for the biomass state being above the desired state, then $v_0^+ > 0$ and $v_0^- \lesseqqgtr 0$. Finally of $q = 0$, then $v_0^+ > 0$ and $v_0^- < 0$. Taking into account these remarks, and given (65)-(66), the following propositions hold.

Proposition 8 (Control uncertainty). *Ceteris Paribus, an increase of the control uncertainties,* $\bar{\sigma}_u, \sigma_u^-, \sigma_u^+$, *will increase the size of the inaction region.*

Thus, the more uncertain is the impact of the control on the state evolution, the wider the Inaction Region should be. In other words, when the control action affects the state evolution with high uncertainty, the best policy is to do nothing up to a considerable drifting of the state away from its desired level. The proof of this proposition is straightforward and comes from the fact that an increase of σ_u^+ and/or σ_u^- will increase the interval for the conditions in (65) to hold. This eases the existence and enlarge the size of the connected region, i.e. the inaction region.

Corollary 2 (No Control Uncertainty). *Without uncertainty on the control effect, the size of inaction region is empty.*

The corollary means that if there is no doubt about the consequences of a change in the TAC policy, then the management rule should always be flexible, accompanying the evolution of $t \to x(t)$.

From (66) it is possible to check that v_0^+ increases with σ_x^+ and $\bar{\sigma}_x$, whereas v_0^- decreases with σ_x^- and $\bar{\sigma}_x$. As a consequence, the conditions $-\bar{\sigma}_u\sigma_u^- \leq x + (v_0^- - r)/2X^-$, for $x < 0$, and the condition $x + (v_0^+ - r)/2X^+ \leq \bar{\sigma}_u\sigma_u^+$, for $x > 0$, in (65) will be less easily verified. As a result, the size of the inaction region will reduce and the following proposition holds.

Proposition 9 (State uncertainty). *An increase of the state uncertainties,* σ_x^+, σ_x^-, *will decrease the size of the inaction region.*

In summary, the more uncertain is the impact of the state level on the state evolution, the less the

Inaction Region may exist. In other words, if doubt exists about the impact of the state evolution, one should intervene as much as possible into the state evolution via the control. Notice that an increase of Q, the cost parameter in the cost function related to the state gap, will have a similar effect to reduce the size of the inaction region (by increasing the value of X^+ and X^-). The consequences of A and α on the size of the inaction region are less obvious since they will impact both X^+ and X^-, and, in an opposite way, v_0^+ or v_0^-.

Finally, there is an obvious proposition which depends on the biomass level being drifted away from the desired level.

Proposition 10 (State deviation). *If the current level of biomass is far away from its desired level* ($|x| \gg 0$)*, distant it will be from a possible Inaction Region.*

5.3 Numerical Simulations

The purpose of this section is to explore some numerical simulations to highlight the previous propositions. In all cases, we set $B = -1$, $R = 0.5$ and $r = -0.1$.

5.3.1 Impact of q and α

The next figures show the range of the inaction region as a function of the q values. In all cases: $A = -0.2$, $\bar{\sigma}_x = \bar{\sigma}_u = 0.2$, $\sigma_x^+ = \sigma_u^+ = 0.3$, $\sigma_x^- = \sigma_u^- = 0.5$. Below the solid line the optimal action u^* is to reduce the TAC ($u^* < 0$, points C and D in in Fig. 2), while above the dashed line, the optimal action is to increase the TAC ($u^* > 0$, points B and E in Fig. 2). Between the two lines lies the inaction region (no change of TAC, $u^* = 0$, point A, Fig. 2).

In Fig. 2, at point A, the state is close to its desired value ($x \simeq 0$) and no reward or penalty is given ($q = 0$). The most appropriate strategy is to do nothing, given the level of uncertainty associated with the use of the control (e.g. uncertainty created by a change of the fixed TAC value). One can see that the position of the inaction region at $q = 0$ is asymmetric around $x = 0$, leading to more room for inaction when $x < 0$ than when $x > 0$. This is explained by the uncertainty values employed in the model for the control, which are asymmetric and it weights more a decrease of u rather than an increase of u, e.g. ${\sigma_u}^+ = 0.3 < \sigma_u^- = 0.5$.

As a result, when the fishery manager is expected to change the fixed TAC, the optimal choice will be not to act so as to avoid uncertainty created by the change of the fixed TAC. Of course, when the deviation from the desired state becomes too important, e.g. $|x| \gg 0$, then the manager should take action whatever the level of uncertainty. Compared to A, points B and C require to act and change

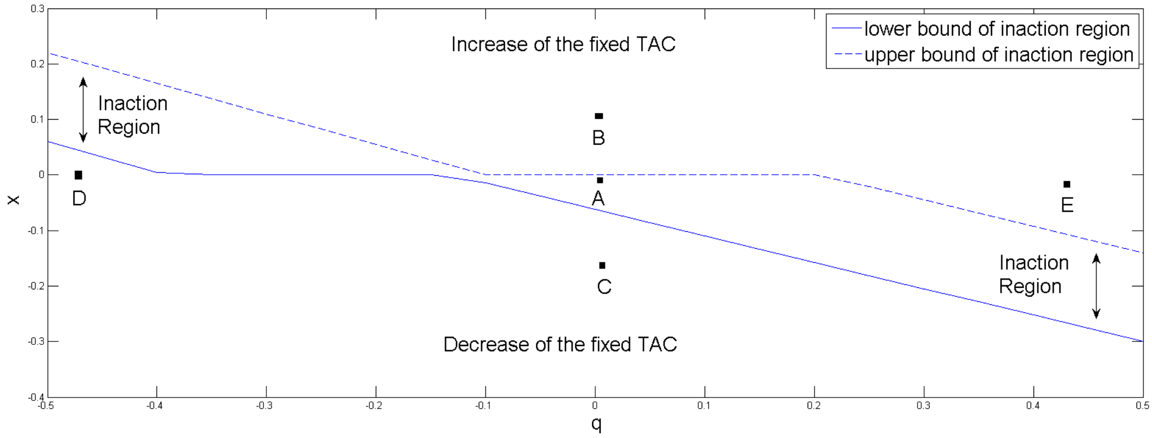

Figure 2: Position of the inaction region in the state x with respect to q, with $\alpha = 0.9$

the rule, either to increase the fixed TAC (point B), or to reduce it (point C). Points D and E have the same state level than A but lead to different interpretations of the situation with respect to the desired state value. In point D, since $q < 0$, there is a reward for the state being above the desired value, e.g. a reward for $x > 0$. In other words, $q < 0$ creates an incentive for the policymaker to set up a reduction of the TAC in order to increase the value of x. On the opposite side, point E with $q > 0$ prevails a penalty for letting the state be above the desired level, e.g. a penalty if $x > 0$. At point E, the policymaker tends to increase the TAC despite of the uncertainty.

More generally, the following remarks apply to Figs. 2–3. Firstly, the position of the inaction region changes with the value of q: a reward for being above the desired state level (e.g. $q < 0$) is more compatible with the problem since the inaction interval arises more for $x > 0$ than for $x < 0$. Secondly, when x drifts away from its desired level, it will leave at some point the inaction region, and the fishery manager should take action by changing the TAC rule. Thirdly, an increase of the discount rate $\alpha = 0.9$ rather than $\alpha = 0.2$ will increase the likelihood of existence of an inaction region but not necessarily its size.

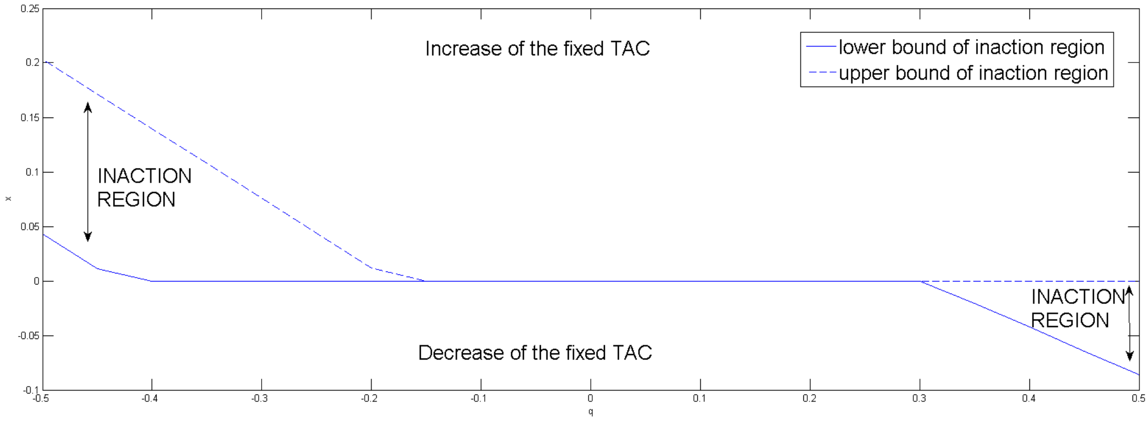

Figure 3: Position of the inaction region in the state x with respect to q, with $\alpha = 0.2$

5.3.2 Impact of control uncertainties

We set in all simulations: $A = -0.2$, $\alpha = 0.9$, $\bar{\sigma}_x = \bar{\sigma}_u = 0.2$, and $\sigma_x^+ = \sigma_x^- = 0$. In this case, we do not assume any endogenous uncertainty generated by the evolution of the state value but only consider the role of control uncertainties. Figs. 4–5 highlight the impact of having asymmetric control uncertainties. Firstly, note that without any state uncertainty, the inaction region has a constant size. Secondly, a comparison of Figs. 4–5 indicates the fact that, *ceteris paribus*, the existence of control uncertainties will shift down (up) the inaction region when uncertainty is mainly due to negative (positive) value of the control, e.g. $\sigma_u^- > 0$ in Fig. 4 (e.g. $\sigma_u^+ > 0$ in Fig. 5). When the inaction region is moving downward (upward), the likelihood of cases where the manager should decrease (increase) the TAC is decreasing.

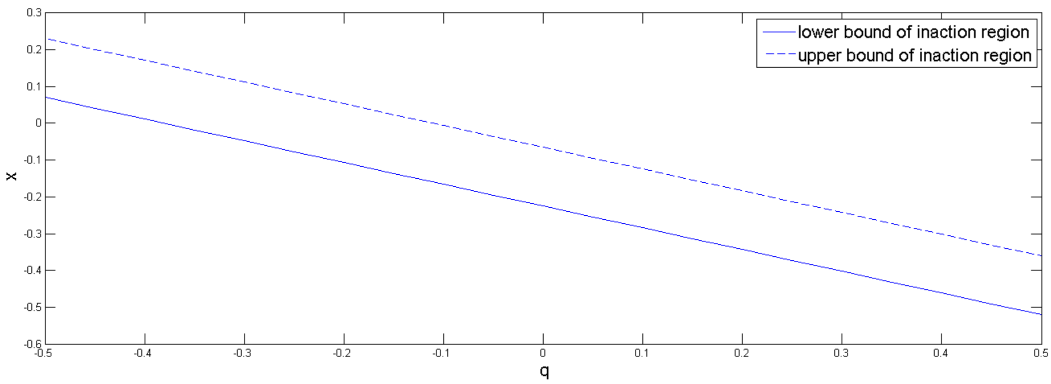

Figure 4: Position of the inaction region in the state x with respect to q, with $\sigma_u^- = 0.8$ and $\sigma_u^+ = 0$

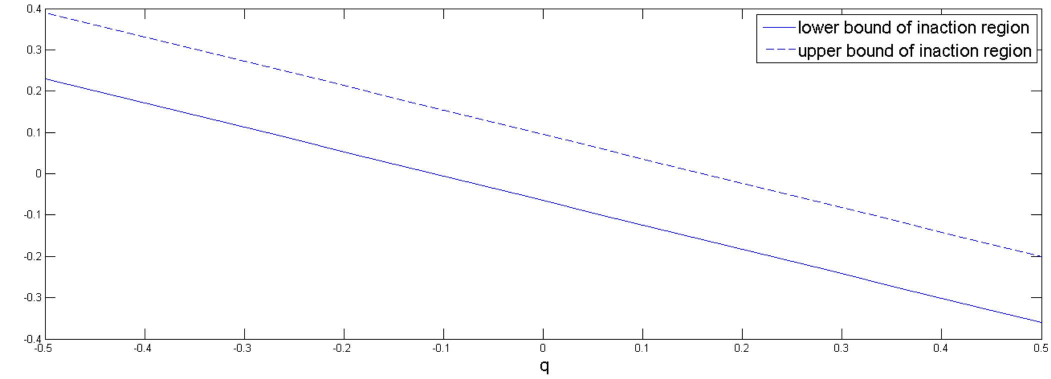

Figure 5: Position of the inaction region in the state x with respect to q, with $\sigma_u^- = 0$ and $\sigma_u^+ = 0.8$

5.3.3 Impact of asymmetric state uncertainties

Figures 6-7 highlight the impact of increasing the state uncertainties given the other uncertainty values. For all simulations, we set: $A = -0.2$, $\alpha = 0.9$, $\bar{\sigma}_x = \bar{\sigma}_u = 0.2$, $\sigma_x^- = \sigma_u^- = 0$, and $\sigma_u^+ = 0.8$. Under this set of parameters, control uncertainty is only created when the policymaker increases the TAC value, that is, when $u^* > 0$. In Fig. 6 one assumes a small positive state uncertainty ($\sigma_x^+ = 0.3$), whereas in Fig. 7 a higher positive state uncertainty value ($\sigma_x^+ = 0.8$) is set. It is seem from the figures that when the level (positive value) of state uncertainty increases, for a given control uncertainty, the inaction region is reduced (Fig. 7). In addition, even when a reward for the state being above the desired level exists, e.g. when $q < 0$, the inaction region moves down (compare with Fig. 6). Since positive values of x create a high level of uncertainty, the best strategy is to reduce x by increasing the TAC.

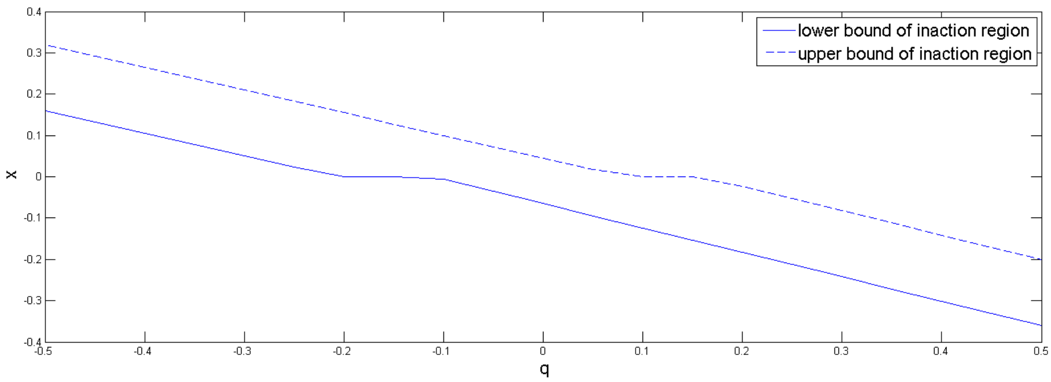

Figure 6: Position of the inaction region in the state x with respect to q, with $\sigma_x^- = \sigma_u^- = 0$ and $\sigma_x^+ = 0.3$ and $\sigma_u^+ = 0.8$

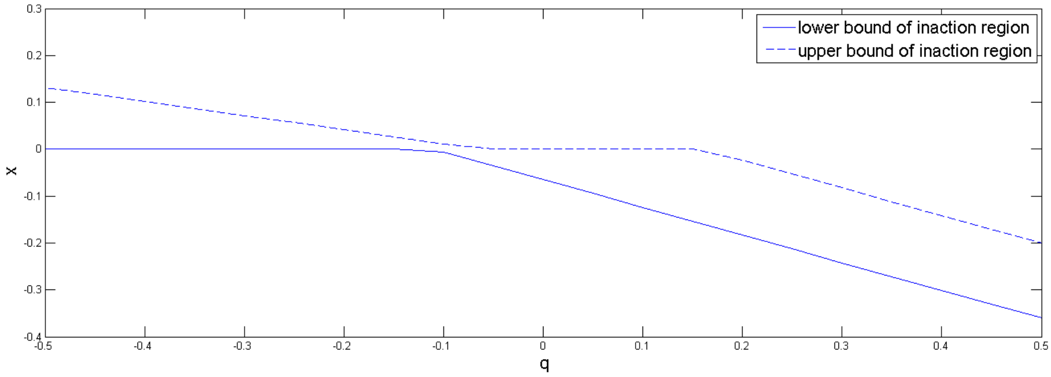

Figure 7: Position of the inaction region in the state x with respect to q, with $\sigma_x^- = \sigma_u^- = 0$ and $\sigma_x^+ = 0.8$ and $\sigma_u^+ = 0.8$

5.3.4 Two biological opposite cases

We conclude the numerical illustrations with a discussion on the desired state value, z_e, and possible consequences of this choice.

K as the desired level

Let assume that $z^e = K$, the carrying capacity. From an ecological perspective, K is a stable equilibrium, which means that we would have $A < 0$. On that basis, there may exists a general agreement between conservationists and fishers that would target an acceptable level of biomass below the K value. Consequently, the fishery manager may include a term $q \geq 0$ in the cost function, which gives a reward for $x < 0$, i.e. $z < K$, and a penalty if $x > 0$, i.e. $z > K$. In Fig. 8, we set $A = -0.1$, with $\alpha = 0.9$, $B = -1$, $\sigma_x^- = \sigma_u^- = 0.2$, $\sigma_x^+ = 0.5$ and $\sigma_u^+ = 0.8$. In words, we assume here that there is more uncertainty when the TAC is decreasing ($u < 0$) and when the biomass is greater than K ($x > 0$), than when the TAC is increasing ($u > 0$), and when the biomass level is less than K ($x < 0$).

The following observations are put forward. At point A, $q = 0$, which means that no reward or penalty of being away from K is implemented, the best option for the manager is to do nothing as long as the biomass is sufficiently close to its desired value K. In points B, C and D, we have $x_B = x_C = x_D = -0.2 < 0$, meaning that the biomass is below the K values. The three points differ because of their q-values: $q_B = 0.1 < q_C = 0.4 < q_D = 0.8$. As q increases, the reward of being below the K value increases. As a consequence, with the same state level, the fishery manager with q_B value

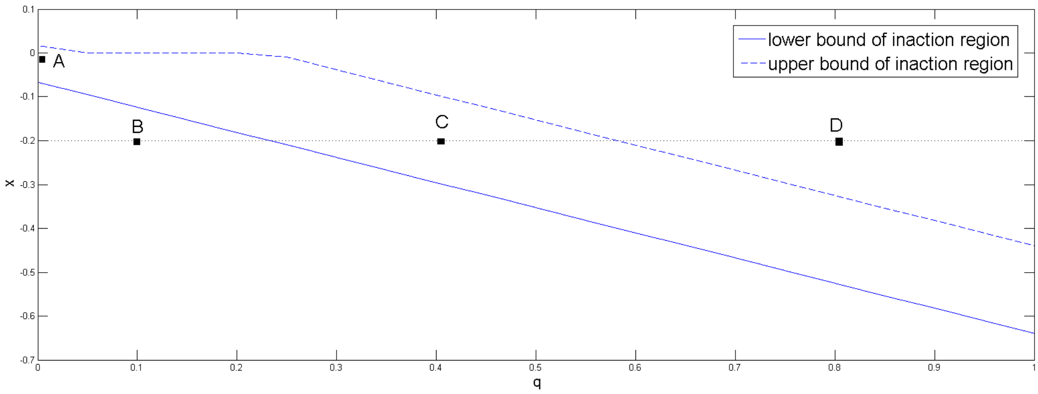

Figure 8: Position of the inaction region in the state x with respect to q, with $\alpha = 0.9$ and with $\sigma_x^- = \sigma_u^- = 0.2$ and $\sigma_x^+ = 0.5$ and $\sigma_u^+ = 0.8$

chooses to reduce the TAC since the biomass is below K, while with another ruler with q_D value would trigger an opposite choice by increasing the TAC. Between these two strategies, a manager using the q_C value would not change the fixed TAC rule.

MSY as the desired level

Let us assume that $z_e = z_{\text{MSY}}$. At the MSY biomass level occurs the maximum growth rate of the population and the growth derivative will always be zero. Thus, $A = 0$ in the model and it is not, per se, an asymptotic equilibrium. Figure 9 shows the inaction region when we set $A = 0$, together with $\alpha = 0.9$ and $B = -1$.

Let us compare situations A, B and C where $x_B = x_C = x_A < 0$. In A, we have $q = 0$, meaning no reward or penalty for being away from the MSY value. At x_A, the biomass level is below the MSY value but belongs to an inaction region, commanding the manager not to act here. Compared to A, in B, we have $q = -0.1$ rewarding any biomass level that would overcome the MSY value, which is desirable from a conservationist viewpoint. In such circumstances, the optimal decision will lead to reduce the TAC. Conversely, in C, the $q = 0.4$ penalty makes it profitable to accept a biomass level below the MSY value, in line with an economic target. The optimal solution will then switch to an increase of the TAC.

Similarly, if the two cases of $q = -0.4$ and $q = 0.4$ were to be compared at $x = 0.025$, an economic-driven strategy will claim for an increase of the TAC, while the conservationist will stick to the fixed TAC rule until the biomass far exceed the MSY level, preferring a sort of precautionary approach.

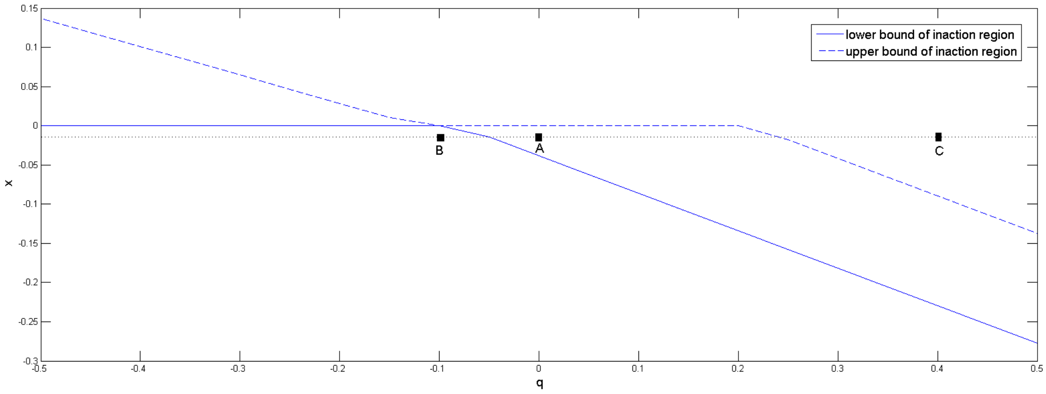

Figure 9: Position of the inaction region in the state x with respect to q, with $\alpha = 0.9$ and with $\sigma_x^- = \sigma_u^- = 0.2$ and $\sigma_x^+ = \sigma_u^+ = 0.5$

6 Discussion and conclusion

The paper develops an asymmetric model to deal with the control problem of continuous stochastic systems with a poorly known dynamics, when the deviations of the desired value should be taken with unbalanced risk. It is based on the CVIU approach that inbuilt in the model the fact that control variations increase the overall state uncertainty. This method is proposed as an alternative to the robust approach in the context of a poor knowledge of the system dynamics.

The use of dynamic programming to design the CVIU model controller requires tools from viscosity analysis, and simple assumptions are made to show that the value function is convex. The optimal feedback control policy is derived and reveals a region in the state space in which the policy is to sustain the action while inside it. Making use of a nomenclature already used in economics, this region is called inaction region, and it creates a precautionary control policy, not seen in the robust worst-case analysis. It is an expected behavior, since the decision maker should exercise care to intervene in a not well known system.

The problem is formulated in an infinite time horizon control framework, as a discounted quadratic cost problem, to study time-invariant optimal solutions. Closed-forms are derived for the solution inside the inaction region, and asymptotically, far away from it. These solutions are quadratic functions obtained as a type of Lyapunov (inside the inaction interval) or Riccati equations (asymptoticaly, as $|x| \to \infty$). Motivated by the fishery management problem, the case of asymmetric uncertainties is studied here for the first time.

We found that the CVIU framework is particularly appropriate in the case of fishery management. It has been recognised that fishery systems dynamics is rather poorly understood, both on the natural and social grounds. The most conventional management measures are based on setting catch (TAC) or effort limits, as implemented by the European Common Fisheries Policy for a number of species (with TACs and quotas fixed every December). However, these management measures rely on accurate stock assessment which is altered by many sources of uncertainties: random shocks caused by environmental variability, data based on catches, errors in parameter estimates, structural uncertainty in ecosystemic models and trophic interactions, inefficient enforcement of harvest quotas resulting in illegal, unregulated and unreported fishing, etc.

According to the source of uncertainty (natural resources facing random oscillations of the environment, stochastic behaviors of fishers responding to harvest control rules), choosing a constant optimal escapement rule is not always possible, as shown in previous studies ([13], [14], [15]). Optimal control becomes even more erratic when the system-to-be-governed includes uncertainty within the model structure itself ([11], [40]). Increasingly, fishery scientists advocate new forms of management policy towards an ecosystemic approach of fisheries, including robust, experimental or adaptive management, harvest control rules, balanced harvest management striving to keep the ecosystem structure and functioning intact, in spite of fishing mortality ([17], [18]). Fishing across ages and sizes and giving up the golden rule of selectivity would enhance the fishery system better than usual recommendations setting an age limit and targeting the older individuals of the fish population. To the extreme, in some cases, the exploited system could be better off with no management at all, avoiding costly and impacting changes of management rules for users ([19], [41]), a position argued by other scientists ([17], [20]).

With a poorly known dynamics of the fishery system, the CVIU approach points out the limit cases within which fishery managers should rather stick to a fixed management rule (e.g. TAC) instead of adapting it permanently to the latest state of knowledge surrounding stock assessment and harvest levels. Following the well-known Brainard principle [21] applied to the implementation of a monetary policy by a central bank, we demonstrate and explain the existence of inaction regions for fishery managers. In this particular case, inaction would not mean for a fishery manager to do nothing at all, but instead to keep a time fixed harvest control rule like a TAC, rather than adjusting it permanently, in view of the inherent uncertainties surrounding the model.

Such inaction regions are more likely to be met when the discount rate is high, the biomass dynamics is stable around the desired level of biomass, and the (state) uncertainty of the biomass decreases. When it comes to uncertainty created by the harvest control rules (e.g. changing the TAC level), we found an asymmetric response of the system dynamics and distinct trends of the inaction region. When the uncertainty coming from harvest control is caused by a decrease of the TAC, the fishery manager ought to adjust the control rule on a regular basis. Conversely, if uncertainty is endogenously created by modifying positively the control rule (i.e. increasing the TAC), then the inaction region is likely to expand and the laissez-faire policy (no new action) should be the rule. Whatever the response of the system to new control rules, whenever the current state of biomass drifts away from the desired (e.g. MSY) level, the inaction region tends to fade away and managers should change the output control (TAC) level.

The implications for fishery management are important. As long as the biomass state level is likely to drift away from the desired level, the risk of critical thresholds and tipping points ([42], [43], [41]) would increase and jeopardize the survival of the population. As long as changing the TAC is not costly, in terms of creating additional uncertainty, the precautionary approach should be imposed in such circumstances and the fishery manager ought to monitor closely the fishery by adjusting downward or upward more frequently the authorized harvest level. On the other hand, if the biomass fluctuates around the desired level of conservation, even in the case of a high state uncertainty, the fishery manager should hold the same control rule and avoid a too frequent intervention. The choice of the desired state level, together with the willingness to reward or penalize more any deviation from this desired state level, will impact on existence and magnitude of the inaction region.

The next step will be to extend the CVIU model in order to better take into account multi-species dynamics, and ecosystem based approach of fishery management. This can be achieved naturally by setting a proper vectorial version of the present model, and by calibrating with data. Finally, a receding horizon solution may be developed to better take into account a rolling planning approach for policy determinations.

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