Green efficiency and Environmental Productivity: Non-parametric Production Analysis

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Abstract

This paper introduces a general framework to analyse green efficiency and environmental productivity. Green productivity and efficiency measures are defined through the new B-disposal scheme. Environmental generalized efficiency measures are introduced to define green productivity indices and indicators. In addition, components of environmental productivity changes are highlighted. New implementation process of environmental efficiency and productivity assessment on convex and non convex B-disposal non-parametric technologies is proposed.

Keywords: B-disposability, Environmental Hicks-Moorsteen index, Environmental Luenberger indicator, Environmental Luenberger-Hicks-Moorsteen indicator, Malmquist-Luenberger index, Non Convexity.

JEL: C61, D24, Q50

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1 Introduction

Deteriorations of global environmental conditions have caused growing interest in environmental efficiency and productivity[1] studies (Sueyoshi et al., 2017; Zhou et al., 2008). Sustainable strategies that allow to reduce impacts on environment are major concerns for private and public sectors. Through their production processes, they attempt to be both environmental responsible and technical efficient. In such a case, managerial efforts to adopt high quality inputs and/or innovative technology to be environmentally efficient are operated.

Traditional eco-efficiency literature is based on the assumption that desirable and undesirable outputs can only be reduced simultaneously by a proportional factor; i.e. weak (or ray) disposal axiom (Shephard, 1970). This modelling of bad output in production theory is due to Färe, Grosskopf, Lovell and Pasurka (1989). Weak Disposal (WD) approach is widely applied to numerous topics in the literature: Manello (2017), Falavigna et al. (2015), Azad and Ancev (2014), Bilsel et al. (2014), Park and Weber (2006) or Picazo-Tadeo et al. (2005). Some recent papers assuming WD models are also proposed in leading journals; see for instance Pham and Zelenyuk (2019). Innovative approaches arose due to the limits of the WD models (Lauwers and Van Huylenbroeck, 2003; Coelli et al., 2007; Lauwers, 2009; Rödseth, 2017; Murty et al., 2012). Dakpo et al. (2016) present a critical review of these recent developments. In the same vein, a more general class of pollution-generating technologies has been defined in Abad and Briec (2019). Through this general framework (convex or not convex), this paper defines innovative environmental generalized efficiency measures. Equivalence conditions among usual green efficiency measures and their generalization are introduced. In addition, this paper shows that generalized eco-efficiency measures allow to define global green efficiency analysis. Indeed, environmental efficiency is defined through various managerial adaptations strategies.

Environmental productivity advances are appraised through different sources (Chung et al., 1997; Sena, 2004; Azad and Ancev 2014; Kapelko et al., 2015; Shen et al., 2017; Dakpo et al., 2019). Knowing green productivity components is of particular interest for firms, policy makers and researchers (Tyteca, 1996; Aiken and Pasurka, 2003; Mahlberg and Sahoo, 2011). Through a novel theoretical framework this paper introduces generalized eco-productivity decomposition. Core components of green productivity growth are defined through convex or non convex environmental technologies (Abad and Briec, 2019). No need to assume the convexity assumption of production technology is major theoretical[2] and empirical implications (De Borger and Kerstens, 1996). Therefore, a global framework to analyse impacts of green investments or environmental policies on green productivity growth components is proposed. In addition, generalized green productivity measures satisfy Diewert and Fox's (2017) essential properties. Hence, this paper allows to relax importance of transitivity or circularity property to establish multilateral or multitemporal comparisons (O'Donnell, 2010, 2012, 2014, 2016) in

[1]Throughout this paper we use similarly the terms environmental efficiency (productivity), green efficiency (productivity) and eco-efficiency (eco-productivity).

[2]Debate of economist's convexities and nature's non convexities is discussed in Dasgupta and Mäler (2003) or Tschirhart (2012).

environmental productivity studies.

The remainder of this paper unfolds as follows. Section 2 introduces technology assumptions and definition. In addition, it highlights environmental distance functions on B-disposal production process. Section 3 defines environmental productivity indices and indicators. These productivity measures inherit the basic structure of the usual productivity indices and indicators. Section 4 proposes new implementation process of environmental efficiency and productivity assessment. Indeed, procedure to implement environmental productivity measures on convex and non convex B-disposal nonparametric technologies is defined. Finally, section 6 discusses and concludes.

2 Environmental technology and distance functions

2.1 Technology assumptions and definition

First, we define the notations used in this paper. Let, $x_t \in \mathbb{R}^n_+$ denotes inputs used to produce no-polluting (desirable) and polluting (undesirable) outputs, $y_t = (y_t^{np}, y_t^p) \in \mathbb{R}^m_+$ with $[m] = [m_{np}] + [m_p]$ where $[m] = card(y_t)$. In addition, assume that $B \subset [m]$ is the subset indexing polluting outputs of the technology[3]. The production possibility set is defined as follows:

$$T_t = \left\{(x_t, y_t^{np}, y_t^p) \in \mathbb{R}^{n+m}_+ : x_t \, \text{can produce} \, (y_t^{np}, y_t^p)\right\} \tag{2.1}$$

The production technology, T_t, can be similarly characterized trough the output set, $P_t : \mathbb{R}^n_+ \longrightarrow 2^{\mathbb{R}^m_+}$, or the input correspondence, $L_t : \mathbb{R}^m_+ \longrightarrow 2^{\mathbb{R}^n_+}$:

$$P_t(x_t) = \left\{(y_t^{np}, y_t^p) \in \mathbb{R}^m_+ : (x_t, y_t^{np}, y_t^p) \in T_t\right\} \tag{2.2}$$

and

$$L_t(y_t^{np}, y_t^p) = \left\{x_t \in \mathbb{R}^n_+ : (x_t, y_t^{np}, y_t^p) \in T_t\right\}. \tag{2.3}$$

Therefore we have necessarily:

$$x_t \in L_t(y_t^{np}, y_t^p) \Leftrightarrow (x_t, y_t^{np}, y_t^p) \in T_t \Leftrightarrow (y_t^{np}, y_t^p) \in P_t(x_t) \tag{2.4}$$

Throughout this paper, we assume that the output correspondence can satisfy the following usual axioms (see Hackman, 2008; Jacobsen, 1970; McFadden, 1978):

$P1$: For all $x_t \in \mathbb{R}^n_+$, $0 \in P_t(x_t)$ and $(y_t^{np}, y_t^p) \notin P_t(0)$, if $(y_t^{np}, y_t^p) \geq 0$ and $(y_t^{np}, y_t^p) \neq 0$; i.e. the inactivity condition holds and there is no free lunch.
$P2$: $P_t(x_t)$ is bounded above for all, $x_t \in \mathbb{R}^n_+$.
$P3$: $P_t(x_t)$ is closed for all, $x_t \in \mathbb{R}^n_+$; from $P2$ and $P3$, $P_t(x_t)$ is a compact.
$P4$: If $v_t \geq x_t \Rightarrow P_t(x_t) \subseteq P_t(v_t)$.
$P5$: $P_t(x_t)$ is a convex set, $\forall x_t \in \mathbb{R}^n_+$.

[3]Note that, if $B = \emptyset$ then, there is no outputs partition. In such case, the outputs are not separated into polluting and no-polluting ones.

In addition to the axioms $P1-P4$, we assume that the outputs satisfy the B-disposal assumption (Abad and Briec, 2019):

$P6$: For all $y^{\emptyset}, y^{B} \in P_t(x_t)$, $y \leq^{\emptyset} y^{\emptyset}$ and $y \leq^{B} y^{B}$ implies $y \in P_t(x_t)$.

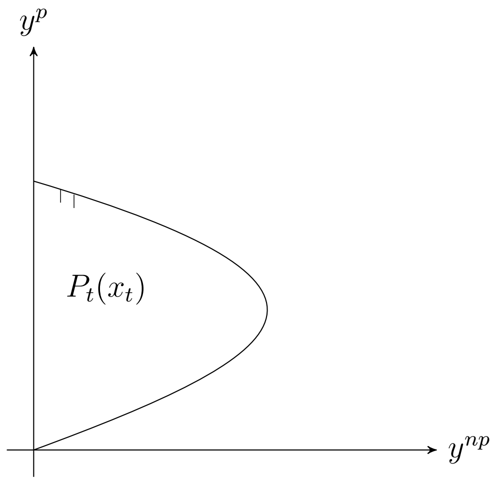

Figure 1: Convex Environmental output set ($P1-P6$)

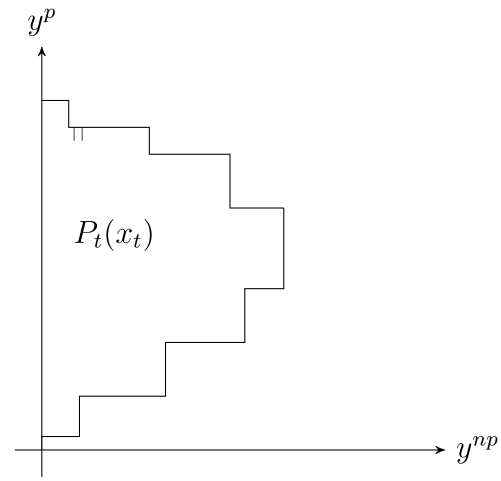

Figure 2: Non convex Environmental output set ($P1-P4$ and $P6$)

Assumptions $P1-P4$ and $P6$ define a general class of environmental output set with traditional strong disposable inputs and B-disposable outputs (polluting and no-polluting; see Figures 1-2). These axioms are fairly weak and do not impose any convexity assumption.

2.2 Environmental efficiency measures

Let us introduce the following convex cone:

$$\mathcal{C}_t^B = \{y_t \in \mathbb{R}^m : y_{t,j} \leq 0 \text{ if } j \in B \text{ and } y_{t,j} \geq 0 \text{ else } \}.$$

For any observations within B-disposal output set, environmental efficiency measures can be defined through the schemes below (Figure 3).

Definition 2.1 *Let $P_t(x_t)$, an environmental output set that satisfies properties $P1-P4$ and $P6$. For any $(x_t, y_t) \in \mathbb{R}_+^{n+m}$, such that $y_t = (y_t^{np}, y_t^{p}) \in P_t(x_t)$, environmental efficiency measures belong to the following subsets:*

i. $\mathcal{S}_1 = (y - \mathbb{R}_+^m) \cap (y + \mathcal{C}_t^B)$,

ii. $\mathcal{S}_3 = (y + \mathbb{R}_+^m) \cap (y + \mathcal{C}_t^B)$ *and*

iii. $\mathcal{S}_2 = (y + \mathcal{C}_t^B) \setminus \{\mathcal{S}_1, \mathcal{S}_3\}$.

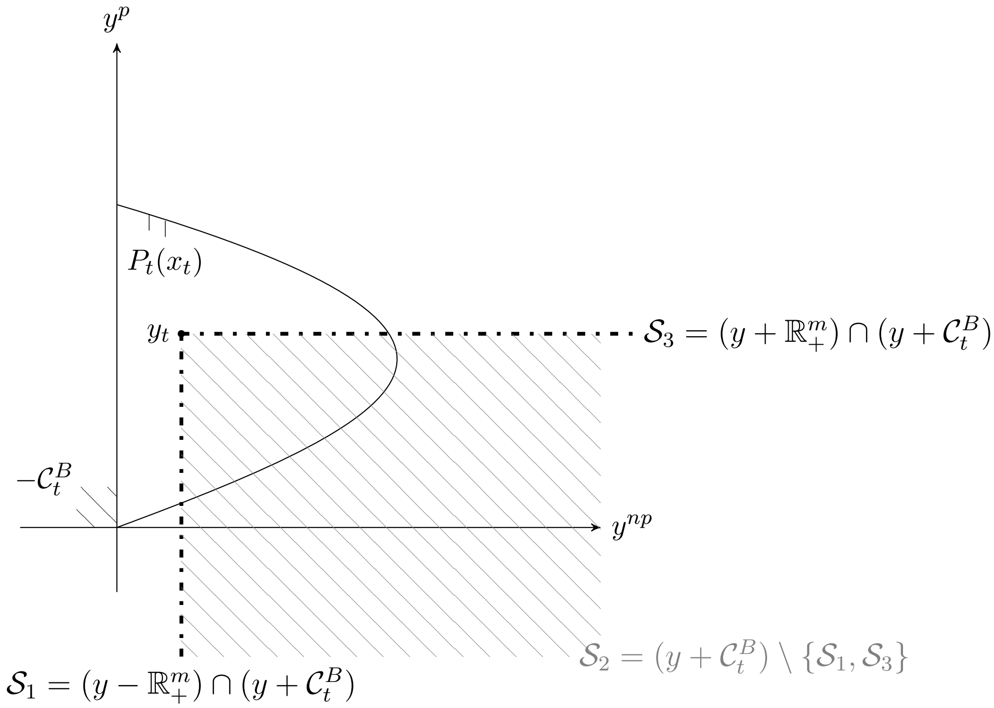

Figure 3: Environmental efficiency analysis ($P1 - P6$)

2.2.1 Multiplicative distance function

Shephard (1953) introduces distance functions that are the inverse of the Debreu-Farrell measures (Debreu, 1951; Farrell, 1957) of technical efficiency. These distance functions can be defined in the input or the output oriented cases. The hyperbolic distance function (Färe et al., 1985) extends Shephard distance functions to the graph of the technology. Distance (or gauge) functions fully characterise technology. Therefore, they have become standard tools for estimating multiplicative measures of technical efficiency.

Through B-disposal output set, the following definition introduces environmental generalized multiplicative distance function.

Definition 2.2 *Let $P_t(x_t)$ be a B-disposal output set that satisfies properties $P1 - P4$. For any $(x_t, y_t) \in \mathbb{R}^{n+m}_+$, such that $y_t = (y^{np}_t, y^p_t) \in \mathbb{R}^m_+$, the environmental generalized multiplicative efficiency measure, $\psi_t : \mathbb{R}^{n+m}_+ \longrightarrow \mathbb{R}^+ \cup \infty$, is defined as follows:*

$$\psi_t(x_t, y_t) = \begin{cases} \inf\limits_{\lambda} \Big\{ \lambda > 0 : & \big(x_t, \lambda^{\beta^p} y^p_t, \lambda^{\beta^{np}} y^{np}_t\big) \in P_t(x_t) \Big\} \\ & \textit{if } \big(x_t, \lambda^{\beta^p} y^p_t, \lambda^{\beta^{np}} y^{np}_t\big) \in P_t(x_t), \lambda > 0 \\ \infty & \textit{else} \end{cases} \tag{2.5}$$

with $\beta^p = \{0, 1\}$ et $\beta^{np} = \{-1, 0\}$.

The following proposition presents equivalence conditions for the environmental generalized multiplicative distance function, desirable (D^{np}_t) and undesirable (D^p_t) output-oriented Shepard distance functions (Färe et al., 2004), and the hyperbolic output (H^o_t) efficiency measure (Färe et al., 1989).

Proposition 2.3 *For any* $(x_t, y_t) \in \mathbb{R}_+^{n+m}$*, such that* $y_t = (y_t^{np}, y_t^p) \in \mathbb{R}_+^m$*, we have:*

i. $\psi_t(x_t, y_t) \equiv D_t^{np}(x_t, y_t)$*, with* $\beta^p = 0$ *and* $\beta^{np} = -1$*.*

ii. $\psi_t(x_t, y_t) \equiv D_t^{p}(x_t, y_t)$*, with* $\beta^p = 1$ *and* $\beta^{np} = 0$*.*

iii. $\psi_t(x_t, y_t) \equiv H_t^{o}(x_t, y_t)$*, with* $\beta^p = 1$ *and* $\beta^{np} = -1$*.*

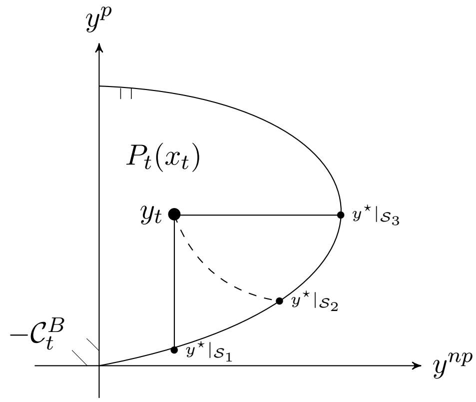

Figure 4: Environmental Generalised Multiplicative Distance Function ($P1 - P6$)

The environmental generalized multiplicative distance function is defined in Figure 4. Distance between points y_t and $y^\star|_{\mathcal{S}_3}$ depicts the desirable output-oriented Shepard distance function (Färe et al., 2004). The gap between points y_t and $y^\star|_{\mathcal{S}_1}$ shows the undesirable output-oriented Shepard distance function (Färe et al., 2004). Finally, distance between points y_t and $y^\star|_{\mathcal{S}_2}$ represents the hyperbolic output efficiency measure (Färe et al., 1989).

2.2.2 Additive distance function

The directional distance function allows for simultaneous input and output variation in the direction of a pre-assigned vector $g_t = (h_t, k_t) \in \mathbb{R}_+^{n+m}$ compatible with the technology (Chambers et al., 1996, 1998)[4]. The special case $g_t = (x_t, y_t)$ is known as the Farrell proportional directional distance function (Briec, 1997) and is a generalization of the Debreu-Farrell efficiency measure[5].

Let us define environmental generalized additive distance function on B-disposal output set.

Definition 2.4 *Let* $P_t(x_t)$ *be an environmental output set that satisfies properties* $P1 - P4$ *and* $P6$*. For all* $(x_t, y_t) \in \mathbb{R}_+^{n+m}$*, such that* $y_t = (y_t^{np}, y_t^p) \in \mathbb{R}_+^m$*, the environmental*

[4]Chambers et al.'s (1996, 1998) directional distance function is an extend of the work of Luenberger (1992, 1995, 1996) in consumer theory.

[5]Axiomatic properties of the proportional directional distance function are defined in Briec (1997) and Chambers, Chung and Färe (1996, 1998).

generalized additive efficiency measure, $\xi^{0,\sigma} : \mathbb{R}_+^{n+m} \times 0 \times [0,1]^{m^{np}} \times [-1,0]^{m^p} \longrightarrow \mathbb{R} \cup -\infty$, *is defined as follows :*

$$\xi_t^{0,\sigma}(x_t, y_t) = \begin{cases} \sup_{\beta} \Big\{ \beta \in \mathbb{R} : \ \Big(x_t, (1+\beta \odot \sigma^{np}) y_t^{np}, (1+\beta \odot \sigma^p) y_t^p\Big) \in P_t(x_t) \Big\} \\ \qquad \qquad \textit{if} \ \Big(x_t, (1+\beta \odot \sigma^{np}) y_t^{np}, (1+\beta \odot \sigma^p) y_t^p\Big) \in P_t(x_t), \beta \in \mathbb{R} \\ \infty \qquad \qquad \textit{else} \end{cases} \tag{2.6}$$

where $\sigma = (\sigma^{np}, \sigma^p) \in [0,1]^{m^{np}} \times [-1,0]^{m^p}$.

The next results establish equivalence relations for the environmental generalized additive efficiency measure, the environmental directional distance function (Chung et al., 1997) and, desirable and undesirable sub-vector directional distance functions (Picazo-Tadeo et al., 2014).

Proposition 2.5 *For any* $(x_t, y_t) \in \mathbb{R}_+^{n+m}$*, such that* $y_t = (y_t^{np}, y_t^p) \in \mathbb{R}_+^m$*, we have:*

i. $\xi_t^{0,\sigma}(x_t, y_t) \equiv \overrightarrow{D}_t^{np}(x_t, y_t; y_t^{np}, 0)$*, with* $\sigma = (\sigma^{np}, \sigma^p) = (1, 0)$.

ii. $\xi_t^{0,\sigma}(x_t, y_t) \equiv \overrightarrow{D}_t^{p}(x_t, y_t; 0, y_t^{p})$*, with* $\sigma = (\sigma^{np}, \sigma^p) = (0, -1)$.

iii. $\xi_t^{0,\sigma}(x_t, y_t) \equiv \overrightarrow{D}_t(x_t, y_t; 0, y_t^{np}, y_t^{p})$*, with* $\sigma = (\sigma^{np}, \sigma^p) = (1, -1)$.

The following corollary presents equivalence conditions for the additive and multiplicative environmental generalized distance functions (Chambers et al., 1996, 1998; Briec, 1997).

Corollary 2.6 *For any* $(x_t, y_t) \in \mathbb{R}_+^{n+m}$*, such that* $y_t = (y_t^{np}, y_t^p) \in \mathbb{R}_+^m$*, we have:*

i. $\xi_t^{0,1,0}(x_t, y_t) \equiv \big[\psi_t^{np}(x_t, y_t)\big]^{-1} - 1$*, where* $\psi_t^{np}(x_t, y_t) = \psi_t(x_t, y_t)$ *such that* $\beta^p = 0$ *and* $\beta^{np} = -1$.

ii. $\xi_t^{0,0,-1}(x_t, y_t) \equiv \big[\psi_t^{p}(x_t, y_t)\big]^{-1} - 1$*, where* $\psi_t^{p}(x_t, y_t) = \psi_t(x_t, y_t)$ *such that* $\beta^p = 1$ *and* $\beta^{np} = 0$.

The environmental generalized additive distance function is defined in Figure 5. Distance between points y_t and $y^\star|_{S3}$ depicts the sub-vector desirable output directional distance function (Picazo-Tadeo et al., 2014). Conversely, the gap between points y_t and $y^\star|_{S1}$ shows the sub-vector undesirable output directional distance function (Picazo-Tadeo et al., 2014). Finally, distance between points y_t and $y^\star|_{S2}$ represents the environmental directional distance function (Chung et al., 1997).

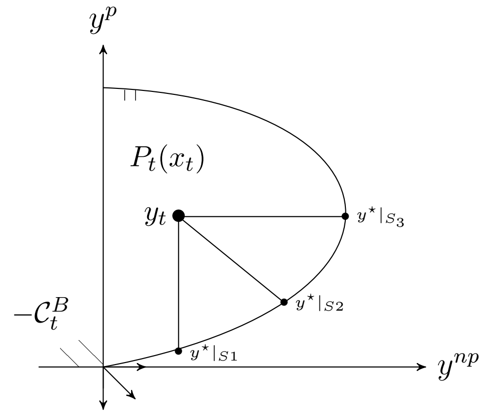

Figure 5: Environmental Generalised Additive Distance Function ($P1 - P6$)

3 Environmental Productivity Indices and Indicators

3.1 Malmquist-Luenberger and Environmental-Luenberger productivity measures

3.1.1 Malmquist-Luenberger productivity index

Chung et al. (1997) define the Malmquist-Luenberger productivity index (ML). Following the environmental generalized efficiency measures, the ML productivity index is defined as follows:

Definition 3.1 *Let $P_t(x_t)$ be a B-disposal output set that satisfies properties $P1 - P4$. For any consecutive time periods $(t, t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, with $y_{t,t+1} = (y^{np}_{t,t+1}, y^{p}_{t,t+1}) \in \mathbb{R}_+^m$, the Malmquist-Luenberger productivity index is defined as follows:*

$$\begin{aligned} &ML_{t,t+1}(x_t, y_t, x_{t+1}, y_{t+1}) \\ &= \left[\frac{1+\xi_t^{0,1,-1}(x_t,y_t)}{1+\xi_t^{0,1,-1}(x_{t+1},y_{t+1})} \times \frac{1+\xi_{t+1}^{0,1,-1}(x_t,y_t)}{1+\xi_{t+1}^{0,1,-1}(x_{t+1},y_{t+1})}\right]^{1/2} \end{aligned} \tag{3.1}$$

When values of the ML index is above (respectively below) unity, then environmental productivity improvement (respectively deterioration) takes place. The ML productivity measure can be decomposed in two components: Efficiency Variation (EV) and Technical Change (TC).

$$MLEV(x_t, y_t, x_{t+1}, y_{t+1}) = \frac{1 + \xi_t^{0,1,-1}(x_t, y_t)}{1 + \xi_{t+1}^{0,1,-1}(x_{t+1}, y_{t+1})} \tag{3.2}$$

and

$$MLTC(x_t, y_t, x_{t+1}, y_{t+1}) = \left[\frac{1 + \xi_{t+1}^{0,1,-1}(x_t, y_t)}{1 + \xi_t^{0,1,-1}(x_t, y_t)} \times \frac{1 + \xi_{t+1}^{0,1,-1}(x_{t+1}, y_{t+1})}{1 + \xi_t^{0,1,-1}(x_{t+1}, y_{t+1})}\right]^{1/2}. \quad (3.3)$$

3.1.2 Environmental-Luenberger productivity indicator

Azad and Ancev (2014) introduce the Environmental Luenberger productivity indicator (EL). This productivity measure identifies output separation (desirable and undesirable) to define environmental productivity growth. The EL productivity indicator is defined as the arithmetic mean of both difference-based Luenberger productivity indicator in base year t (first difference) and $t+1$ (second difference)[6]. For a B-disposal output set, the EL productivity measure is defined as follows:

Definition 3.2 *Let $P_t(x_t)$ be an environmental output set that satisfies properties $P1-P4$ and $P6$. For any consecutive time periods $(t, t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, with $y_{t,t+1} = (y_{t,t+1}^{np}, y_{t,t+1}^{p}) \in \mathbb{R}_+^m$, the Environmental Luenberger productivity indicator is defined as follows:*

$$\begin{aligned} & EL_{t,t+1}(x_t, x_{t+1}, y_t, y_{t+1}) \\ & = \tfrac{1}{2} \Big[\Big(\xi_t^{0,1,-1}(x_t, y_t) - \xi_t^{0,1,-1}(x_{t+1}, y_{t+1}) \Big) \\ & + \Big(\xi_{t+1}^{0,1,-1}(x_t, y_t) - \xi_{t+1}^{0,1,-1}(x_{t+1}, y_{t+1}) \Big) \Big] \end{aligned} \quad (3.4)$$

The EL productivity indicator highlights environmental improvement, respectively decline, when it takes positive, respectively negative, values. The environmental Luenberger productivity indicator can be decomposed as follows:

$$\begin{aligned} & EL_{t,t+1}(x_t, x_{t+1}, y_t, y_{t+1}) \\ & = \left[\xi_t^{0,1,-1}(x_t, y_t) - \xi_{t+1}^{0,1,-1}(x_{t+1}, y_{t+1})\right] \\ & + \tfrac{1}{2} \Big[\Big(\xi_{t+1}^{0,1,-1}(x_{t+1}, y_{t+1}) - \xi_t^{0,1,-1}(x_{t+1}, y_{t+1}) \Big) \\ & + \Big(\xi_{t+1}^{0,1,-1}(x_t, y_t) - \xi_t^{0,1,-1}(x_t, y_t) \Big) \Big] \end{aligned} \quad (3.5)$$

Difference in the first bracket shows Efficiency Variation (EV) over time $(t, t+1)$. Terms in the second bracket define Technical Variation (TV) over periods (t) and $(t+1)$.

3.1.3 Environmental cross-time efficiency measure: infeasibility issue

Let us consider the following cross-time additive efficiency measure.

Definition 3.3 *For any consecutive time periods $(t, t+1)$, let $P_t(x_t)$ be a B-disposal output set that satisfies properties $P1-P4$. For any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, with $y_{t,t+1} =$*

[6] Picazo-Tadeo et al. (2014) adopt the same approach. However, sub-vector output directional distance functions are used.

$(y^{np}_{t,t+1}, y^{p}_{t,t+1}) \in \mathbb{R}^m_+$, *the environmental cross-time additive efficiency measure is defined as follows:*

$$\xi_l^{0,1,-1}(x_s, y_s) = \begin{cases} \sup_{\beta} \left\{ \beta \in \mathbb{R} : \quad \left(x_s, (1+\beta) y_s^{np}, (1-\beta) y_s^p \right) \in P_l(x_l) \right\} \\ \qquad \textit{if} \left(x_s, (1+\beta) y_s^{np}, (1-\beta) y_s^p \right) \in P_l(x_l), \beta \in \mathbb{R} \\ \infty \qquad \textit{else} \end{cases} \tag{3.6}$$

where $s, l = t, t+1$ *with* $s \neq l$.

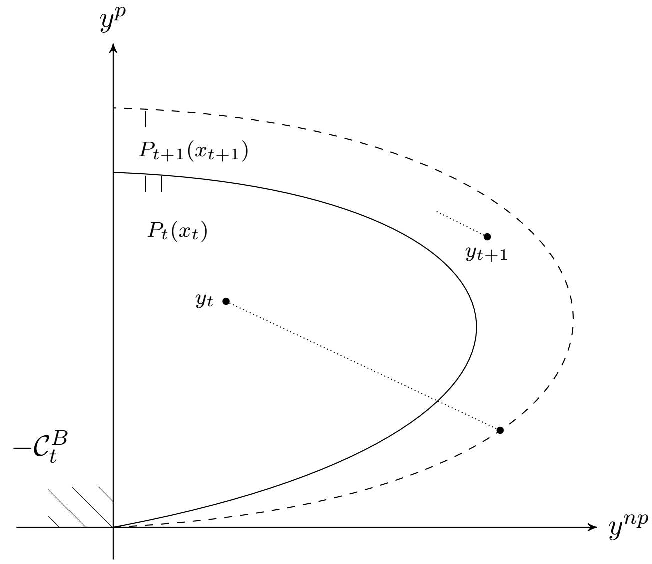

Figure 6: Environmental Generalised Additive Distance Function $(t, t+1)$

In Figure 6, we have $P_t(x_t) \subset P_{t+1}(x_{t+1})$ and $y_{t+1} \in P_{t+1}(x_{t+1})$. In addition, the cross-time distance function $\xi_t^{0,1,-1}(x_{t+1}, y_{t+1}) = \infty$. In such case, ML and EL technical change components do not have finite value. Therefore, $EL_{t,t+1}(x_t, x_{t+1}, y_t, y_{t+1}) = ML_{t,t+1}(x_t, x_{t+1}, y_t, y_{t+1}) = \infty$.

ML and EL infeasibility issues are major concerns for firms, policy makers or researchers interested in empirical environmental productivity studies. Indeed, in such a case it is not possible to define global green growth productivity analysis (Abad and Ravelojaona, 2017).

3.2 Environmental Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures

3.2.1 Environmental Hicks-Moorsteen productivity index

Abad (2015) defines environmental productivity measure that inherits basic structure of the Hicks-Moorsteens productivity index. Environmental Hicks-Moorsteens (EHM) index is defined as the ratio of Malmquist good output quantity index (EM_t^{np}) and Malmquist bad output quantity index (EM_t^p). For a B-disposal output set, the EHM productivity measure is defined as follows:

Definition 3.4 *Let $P_t(x_t)$ be an environmental output set that satisfies properties $P1-P4$ and $P6$. For any consecutive time periods $(t, t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, with $y_{t,t+1} = (y_{t,t+1}^{np}, y_{t,t+1}^{p}) \in \mathbb{R}_+^m$, the environmental Hicks-Moorsteens index for time period (t) is defined as follows:*

$$EHM_t(x_t, y_t^{np}, y_t^p, y_{t+1}^{np}, y_{t+1}^p) = \frac{EM_t^{np}(x_t, y_t^{np}, y_{t+1}^{np}, y_t^p)}{EM_t^p(x_t, y_t^{np}, y_t^p, y_{t+1}^p)}. \tag{3.7}$$

Where EM_t^{np} and EM_t^p are respectively no polluting and polluting Malmquist quantity indices for time period (t).

No polluting and polluting Malmquist quantity indices for time period (t) are respectively defined by:

$$EM_t^{np}(x_t, y_t^{np}, y_{t+1}^{np}, y_t^p) = \frac{\psi_t^{np}(x_t, y_{t+1}^{np}, y_t^p)}{\psi_t^{np}(x_t, y_t^{np}, y_t^p)} \tag{3.8}$$

and

$$EM_t^p(x_t, y_t^{np}, y_t^p, y_{t+1}^p) = \frac{\psi_t^p(x_t, y_t^{np}, y_{t+1}^p)}{\psi_t^p(x_t, y_t^{np}, y_t^p)}. \tag{3.9}$$

No polluting Malmquist quantity index is always greater (lesser) than unity when, no polluting outputs increase (decline) among the time periods $(t, t+1)$, for given polluting outputs and inputs. In addition, when the polluting Malmquist quantity index is lesser (greater) than unity then, polluting outputs decrease (increase) among the time periods $(t, t+1)$, for given no polluting outputs and inputs. Therefore, when EHM index is greater, respectively lesser, than unity, then it highlights environmental productivity improvement, respectively deterioration.

Following consecutive time periods $(t, t+1)$, global output EHM index is defined as the geometric mean of output environmental Hicks-Moorsteen indices for time period (t) and $(t+1)$.

Proposition 3.5 *Let $P_t(x_t)$ be a B-disposal output set that satisfies properties $P1-P4$. For any consecutive time periods $(t, t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, with $y_{t,t+1} = (y_{t,t+1}^{np}, y_{t,t+1}^{p}) \in \mathbb{R}_+^m$, the global environmental Hicks-Moorsteen index is defined as follows:*

$$\begin{aligned} &EHM_{t,t+1}(x_t, y_t^{np}, y_t^p, y_{t+1}^{np}, y_{t+1}^p) = \\ &\left[EHM_t(x_t, y_t^{np}, y_t^p, y_{t+1}^{np}, y_{t+1}^p) \times EHM_{t+1}(x_t, y_t^{np}, y_t^p, y_{t+1}^{np}, y_{t+1}^p)\right]^{1/2}. \end{aligned} \tag{3.10}$$

3.2.2 Environmental Luenberger-Hicks-Moorsteen productivity indicator

Abad (2015) introduces an environmental productivity indicator that inherits the basic structure of the Luenberger-Hicks-Moorsteen productivity indicator. Environmental Luenberger-Hicks-Moorsteen (ELHM) indicator is defined as the difference between environmental Luenberger good output quantity indicator (EL_t^{np}) and environmental Luenberger bad output quantity indicator (EL_t^p). For a B-disposal output set, the ELHM productivity measure is defined as follows:

Definition 3.6 *Let $P_t(x_t)$, an environmental output set that satisfies properties $P1-P4$ and $P6$. For any consecutive time periods $(t, t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, with $y_{t,t+1} = (y_{t,t+1}^{np}, y_{t,t+1}^{p}) \in \mathbb{R}_+^m$, the environmental Luenberger-Hicks-Moorsteen indicator for time period (t) is defined as follows:*

$$\begin{aligned} & ELHM_t(x_t, y_t^{np}, y_t^p, y_{t+1}^{np}, y_{t+1}^p) \\ & = EL_t^{np}(x_t, y_t^{np}, y_t^p, y_{t+1}^{np}) - EL_t^p(x_t, y_t^{np}, y_t^p, y_{t+1}^p), \end{aligned} \tag{3.11}$$

Where EL_t^{np} and EL_t^p are respectively no polluting and polluting Luenberger quantity indicators for time period (t).

No polluting and polluting Luenberger quantity indicators for time period (t) are defined as follows:

$$EL_t^{np}(x_t, y_t^{np}, y_t^p, y_{t+1}^{np}) = \xi_t^{0,1,0}(x_t, y_t) - \xi_t^{0,1,0}(x_t, y_{t+1}^{np}, y_t^p) \tag{3.12}$$

and

$$EL_t^p(x_t, y_t^{np}, y_t^p, y_{t+1}^p) = \xi_t^{0,0,-1}(x_t, y_t^{np}, y_{t+1}^p) - \xi_t^{0,0,-1}(x_t, y_t). \tag{3.13}$$

When no polluting Luenberger quantity indicator is greater (respectively smaller) than zero, more (respectively less) desirable outputs are produced in period $(t+1)$ than in period (t) for given undesirable output and input vectors. Polluting Luenberger quantity indicator is smaller (respectively greater) than zero if less (respectively more) bad outputs are generated in period $(t+1)$ than in period (t) for given input and good output vectors. Therefore, ELHM productivity indicator exhibits environmental productivity improvement (deterioration) when it takes positive (negative) values.

For any consecutive time periods $(t, t+1)$, global ELHM indicator is defined as the arithmetic mean of time periods (t) and $(t+1)$ environmental Luenberger-Hicks-Moorsteen indicators.

Proposition 3.7 *Let $P_t(x_t)$ be a B-disposal output set that satisfies properties $P1-P4$. For any consecutive time periods $(t, t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, with $y_{t,t+1} = (y_{t,t+1}^{np}, y_{t,t+1}^{p}) \in \mathbb{R}_+^m$, the global environmental Luenberger-Hicks-Moorsteen indicator is defined as follows:*

$$\begin{gathered} ELHM_{t,t+1}(x_t, y_t^{np}, y_t^p, y_{t+1}^{np}, y_{t+1}^p) = \\ \frac{1}{2}\big[ELHM_t(x_t, y_t^{np}, y_t^p, y_{t+1}^{np}, y_{t+1}^p) + ELHM_{t+1}(x_t, y_t^{np}, y_t^p, y_{t+1}^{np}, y_{t+1}^p)\big]. \end{gathered} \tag{3.14}$$

3.2.3 Global environmental productivity analysis

Let us consider the following sub-vector cross-time additive efficiency measures.

Definition 3.8 *For any consecutive time periods $(t, t+1)$, let $P_t(x_t)$ be an environmental output set that satisfies properties $P1-P4$ and $P6$. For any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, with*

$y_{t,t+1} = (y^{np}_{t,t+1}, y^p_{t,t+1}) \in \mathbb{R}^m_+$, *no polluting and polluting cross-time additive efficiency measures are respectively defined as follows:*

$$\xi_l^{0,1,0}(x_s,y_s) = \begin{cases} \sup\limits_{\beta}\left\{\beta \in \mathbb{R}: \quad \left(x_s,(1+\beta)y_s^{np},y_s^p\right) \in P_l(x_l)\right\} \\ \qquad\qquad \text{if } \left(x_s,(1+\beta)y_s^{np},y_s^p\right) \in P_l(x_l), \beta \in \mathbb{R} \\ \infty \qquad\qquad \text{else} \end{cases} \tag{3.15}$$

and

$$\xi_l^{0,0,-1}(x_s,y_s) = \begin{cases} \sup\limits_{\beta}\left\{\beta \in \mathbb{R}: \quad \left(x_s,y_s^{np},(1-\beta)y_s^p\right) \in P_l(x_l)\right\} \\ \qquad\qquad \text{if } \left(x_s,y_s^{np},(1-\beta)y_s^p\right) \in P_l(x_l), \beta \in \mathbb{R} \\ \infty \qquad\qquad \text{else} \end{cases} \tag{3.16}$$

where $s, l = t, t+1$ *with* $s \neq l$.

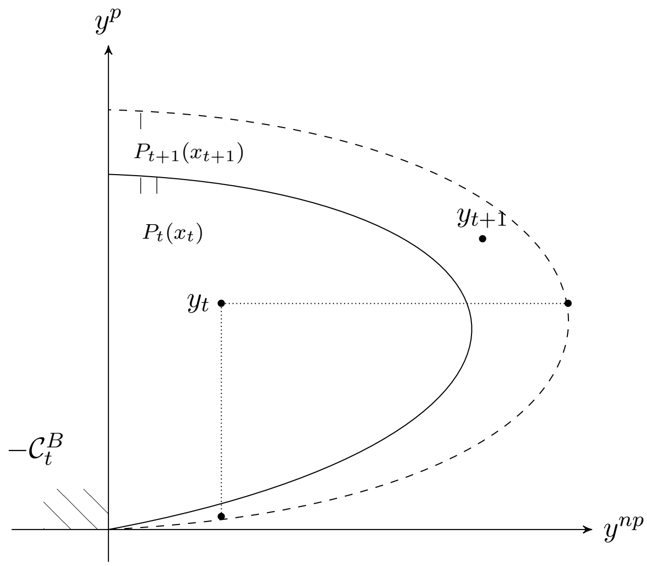

Figure 7: Environmental Generalised Additive Distance Function $(t, t+1)$

For any consecutive time periods $(t, t+1)$, proposition below suggests that the global EHM and ELHM productivity measures are always properly defined.

Proposition 3.9 *Let $P_t(x_t)$ be a B-disposal output set that satisfies properties $P1-P4$. For any consecutive time periods $(t, t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}^{n+m}_+$, such that $y_t = (y_t^{np}, y_t^p) \in P_t(x_t)$ and $y_{t+1} = (y_{t+1}^{np}, y_{t+1}^p) \in P_{t+1}(x_{t+1})$, the global EHM and ELHM productivity measures always have finite value.*

Proof: Assume that $y_s = (y_s^{np}, y_s^p) \in P_s(x_s)$, with $s = t, t+1$. Therefore, $\xi_s^{0,1,0}(x_s,y_s) \in (y_s + \mathbb{R}^m_+) \cap (y_s + \mathcal{C}_s^B)$. Consequently,

i. $\xi_s^{0,1,0}(x_s,y_s) = 0$ if $P_s(x_s) \setminus (y_s + \mathbb{R}^m_+) \cap (y_s + \mathcal{C}_s^B) = \emptyset$ and

ii. $\xi_s^{0,1,0}(x_s, y_s) > 0$ else.

In addition, if observations y^{np} and y^p are merged in t and $t+1$ to form fictive observation $y^f = (y_l^{np}, y_s^p)$ with $s, l = t, t+1$ and $s \neq l$, then:

iii. $\xi_s^{0,1,0}(x_s, y^f) = 0$ if $P_s(x_s) \setminus (y^f + \mathbb{R}_+^m) \cap (y^f + \mathcal{C}_s^B) = \emptyset$ and

iv. $\xi_s^{0,1,0}(x_s, y^f) \gtreqless 0$ else.

The same applies to the sub-vector distance function $\xi_l^{0,0,-1}(x_s, y_s)$. Following Corollary 2.6, this ends the proof. □

The above result is of particular interest for firms, policy makers or researchers to define global environmental productivity recommendations. Indeed, proposition 4.4 shows that EHM or ELHM productivity measures are always feasible.

4 Decomposition of environmental Luenberger-Hicks-Moorsteen and Hicks-Moorsteen productivity measures

4.1 Environmental Luenberger-Hicks-Moorsteen productivity indicator

Additive decomposition (Ang and Kerstens, 2017) of the environmental Luenberger-Hicks-Moorsteen productivity indicator is defined below.

Definition 4.1 *Let $P_t(x_t)$ be a B-disposal output set that satisfies properties $P1 - P4$. For any consecutive time periods $(t, t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, with $y_{t,t+1} = (y_{t,t+1}^{np}, y_{t,t+1}^p) \in \mathbb{R}_+^m$, the global environmental Luenberger-Hicks-Moorsteens indicator can be decomposed as follows:*

$$ELHM_{t,t+1} = ETC_{t,t+1} + EEV_{t,t+1} + ESEC_{t,t+1} \tag{4.1}$$

Where,

i. *$ETC_{t,t+1}$ shows Environmental Technical Change among the periods (t) and $(t+1)$.*

ii. *$EEV_{t,t+1}$ depicts Environmental Efficiency Variation among the periods (t) and $(t+1)$.*

iii. *$ESEC_{t,t+1}$ allows to identify Environmental Scale Efficiency Change between the periods (t) and $(t+1)$.*

Environmental technical change between time periods (t) and $(t+1)$ is defined as follows (Figure 8):

$$ETC_{t,t+1} = TC_{t,t+1}^{np} + TC_{t,t+1}^p. \tag{4.2}$$

Where,

$$TC^{np}_{t,t+1} = \frac{1}{2}\Big[\big(\xi^{0,1,0}_{t+1}(x_{t+1}, y^{np}_{t}, y^{p}_{t+1}) - \xi^{0,1,0}_{t}(x_t, y^{np}_t, y^p_t)\big) + \big(\xi^{0,1,0}_{t+1}(x_{t+1}, y^{np}_{t+1}, y^{p}_{t+1}) - \xi^{0,1,0}_{t}(x_t, y^{np}_{t+1}, y^p_t)\big)\Big] \quad (4.3)$$

$$TC^{p}_{t,t+1} = \frac{1}{2}\Big[\big(\xi^{0,0,-1}_{t+1}(x_{t+1}, y^{np}_{t+1}, y^{p}_{t}) - \xi^{0,0,-1}_{t}(x_t, y^{np}_t, y^p_t)\big) + \big(\xi^{0,0,-1}_{t+1}(x_{t+1}, y^{np}_{t+1}, y^{p}_{t+1}) - \xi^{0,0,-1}_{t}(x_t, y^{np}_{t}, y^p_{t+1})\big)\Big] \quad (4.4)$$

show technical change in no polluting and polluting outputs directions over periods (t) and $(t+1)$. $TC^{np}_{t,t+1}$ ($TC^{p}_{t,t+1}$) presents no polluting (polluting) technical variation for observation (y^{np}_t, y^p_t) assessed with respect to $(t+1)$'s and (t)'s production frontier. When $TC^{np}_{t,t+1} > 0$ ($TC^{p}_{t,t+1} > 0$) then, B-disposal boundary shifts to the right (downwards). Therefore, $ETC_{t,t+1}$ evaluates technical change in both polluting and no polluting directions. When $ETC_{t,t+1} > 0$ then, global environmental technical progress occurs. The combination of the measures of $TC^{np}_{t,t+1}$ and $TC^{p}_{t,t+1}$ offers informations about the variation of the production technology. Table 1 summarizes the conditions of environmental technical change characterization.

	$TC^{p}_{t,t+1} > 0$	$TC^{p}_{t,t+1} < 0$
$TC^{np}_{t,t+1} > 0$	$ETC_{t,t+1} > 0$, shift to the right and to the downwards	**i.** $\lvert TC^{np}_{t,t+1}\rvert > \lvert TC^{p}_{t,t+1}\rvert$ then $ETC_{t,t+1} > 0$, shift to the right and to the upwards **ii.** $\lvert TC^{np}_{t,t+1}\rvert < \lvert TC^{p}_{t,t+1}\rvert$ then $ETC_{t,t+1} < 0$, shift to the right and to the upwards
$TC^{np}_{t,t+1} < 0$	**i.** $\lvert TC^{np}_{t,t+1}\rvert < \lvert TC^{p}_{t,t+1}\rvert$ then $ETC_{t,t+1} > 0$, shift to the left and to the downwards **ii.** $\lvert TC^{np}_{t,t+1}\rvert < \lvert TC^{p}_{t,t+1}\rvert$ then $ETC_{t,t+1} < 0$, shift to the left and to the downwards	$ETC_{t,t+1} < 0$, shift to the left and to the upwards

Table 1: Characterization of environmental technical change

Environmental efficiency variation among time periods (t) and $(t+1)$ is defined as follows (Figure 9):

$$EEV_{t,t+1} = EV^{np}_{t,t+1} + EV^{p}_{t,t+1}. \quad (4.5)$$

Where,

$$EV^{np}_{t,t+1} = \xi^{0,1,0}_{t}(x_t, y^{np}_t, y^p_t) - \xi^{0,1,0}_{t+1}(x_{t+1}, y^{np}_{t+1}, y^p_{t+1}) \quad (4.6)$$

and

$$EV^{p}_{t,t+1} = \xi^{0,0,-1}_{t}(x_t, y^{np}_t, y^p_t) - \xi^{0,0,-1}_{t+1}(x_{t+1}, y^{np}_{t+1}, y^p_{t+1}) \quad (4.7)$$

are respectively no polluting and polluting efficiency variation among periods (t) and $(t+1)$. When $EV^{np}_{t,t+1} > 0$ ($EV^{p}_{t,t+1} > 0$) then, efficiency increases in no polluting (polluting) direction over time $(t, t+1)$. Therefore, $EEV_{t,t+1} > 0$ shows global environmental efficiency improvement among periods (t) and $(t+1)$. Table 2 summarizes the conditions of environmental efficiency variation.

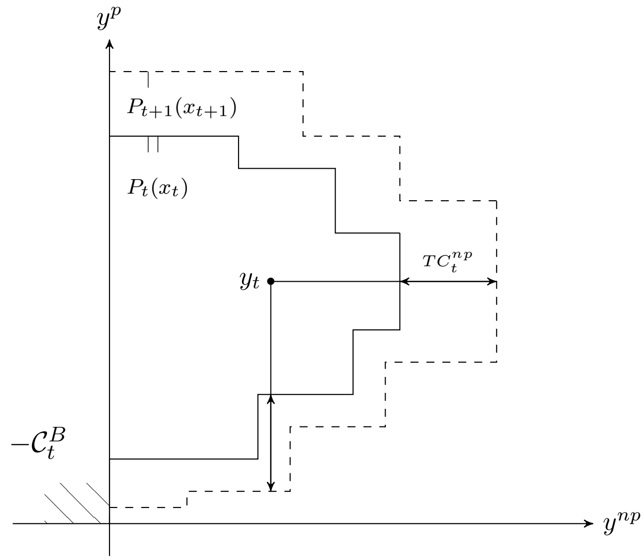

Figure 8: Technological change of non convex output set $(t, t+1)$

	$EV^p_{t,t+1} > 0$	$EV^p_{t,t+1} < 0$
$EV^{np}_{t,t+1} > 0$	$EEV_{t,t+1} > 0$	**i.** $\lvert EV^{np}_{t,t+1}\rvert > \lvert EV^p_{t,t+1}\rvert$ then $EEV_{t,t+1} > 0$, **ii.** $\lvert EV^{np}_{t,t+1}\rvert < \lvert EV^p_{t,t+1}\rvert$ then $EEV_{t,t+1} < 0$
$EV^{np}_{t,t+1} < 0$	**i.** $\lvert EV^{np}_{t,t+1}\rvert < \lvert EV^p_{t,t+1}\rvert$ then $EEV_{t,t+1} > 0$ **ii.** $\lvert EV^{np}_{t,t+1}\rvert < \lvert EV^p_{t,t+1}\rvert$ then $EEV_{t,t+1} < 0$	$EEV_{t,t+1} < 0$

Table 2: Characterization of environmental efficiency variation

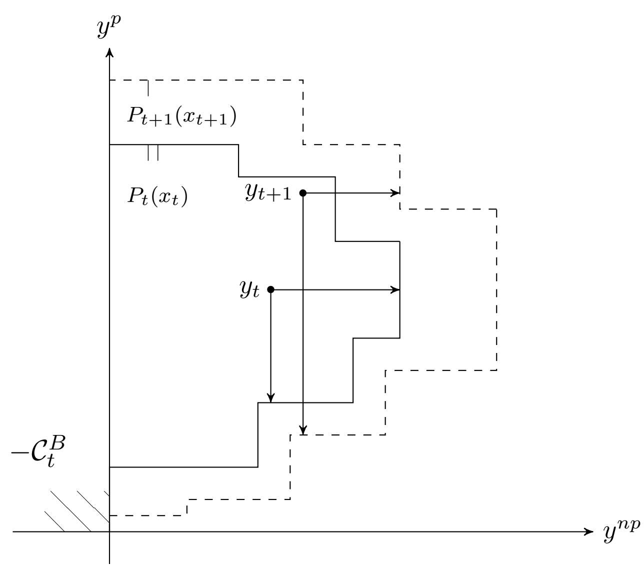

Figure 9: Technical efficiency change of non convex output set $(t, t+1)$

Finally, environmental scale efficiency change among periods (t) and $(t+1)$ is defined as :

$$ESEC_{t,t+1} = ELHM_{t,t+1} - ETC_{t,t+1} - EEV_{t,t+1} \tag{4.8}$$

Therefore, residual resulting from the difference between ELHM, ETC and EEV, coincide to environmental scale efficiency change (Ang and Kerstens, 2017; Diewert and Fox, 2017). If there are no technical change ($ETC_{t,t+1} = 0$) and no technical inefficiency variation ($EEV_{t,t+1} = 0$) then, environmental productivity growth comes from scale efficiency change ($ESEC_{t,t+1} = ELHM_{t,t+1}$). This result is consistent with Diewert and Fox's (2017) approach. For any consecutive time periods $(t, t+1)$, if the B-disposal output set does not change (no technological variation) and if the firm is technically efficient (no efficiency variation) then, firm's productivity can change by moving along the B-disposal output set boundary (Balk, 2001).

Additive decomposition of environmental scale efficiency variation is defined below (Ang and Kerstens, 2017; Diewert and Fox, 2017); see Figure 10.

Proposition 4.2 *Let $P_t(x_t)$ be a B-disposal output set that satisfies properties $P1-P4$. For any consecutive time periods $(t, t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, such that $y_t = (y_t^{np}, y_t^p) \in P_t(x_t)$ and $y_{t+1} = (y_{t+1}^{np}, y_{t+1}^p) \in P_{t+1}(x_{t+1})$, additive decomposition of scale efficiency change is defined as follows:*

$$ESEC_{t,t+1} = \frac{1}{2}\left(ESEC_t + ESEC_{t+1}\right). \tag{4.9}$$

Where $ESEC_t$ ($ESEC_{t+1}$) shows the change in desirable and undesirable outputs along the (t)'s ($(t+1)$'s) B-disposal frontier.

No polluting and polluting outputs variation along the (t)'s B-disposal frontier is defined as follows:

$$ESEC_t = SEC_t^{np} + SEC_t^p. \tag{4.10}$$

With,

$$SEC_t^{np} = \xi_t^{0,1,0}(x_t, y_{t(t+1)}^{np^\star}, y_t^p) - \xi_t^{0,1,0}(x_t, y_t^{np^\star}, y_t^p) \tag{4.11}$$

and

$$SEC_t^p = \xi_t^{0,0,-1}(x_t, y_t^{np}, y_t^{p^\star}) - \xi_t^{0,0,-1}(x_t, y_t^{np}, y_{t(t+1)}^{p^\star}). \tag{4.12}$$

Desirable and undesirable outputs optimal projection along the (t)'s B-disposal boundary are respectively defined as follows:

$$y_t^{np^\star} = \left(1 + \xi_t^{0,1,0}(x_t, y_t^{np}, y_t^p)\right) y_t^{np} \tag{4.13}$$
$$y_t^{p^\star} = \left(1 + \xi_t^{0,0,-1}(x_t, y_t^{np}, y_t^p)\right) y_t^p \tag{4.14}$$

and

$$y_{t(t+1)}^{np^\star} = \left(1 + \xi_t^{0,1,0}(x_{t+1}, y_{t+1}^{np}, y_{t+1}^p)\right) y_{t+1}^{np} \tag{4.15}$$
$$y_{t(t+1)}^{p^\star} = \left(1 + \xi_t^{0,0,-1}(x_{t+1}, y_{t+1}^{np}, y_{t+1}^p)\right) y_{t+1}^p. \tag{4.16}$$

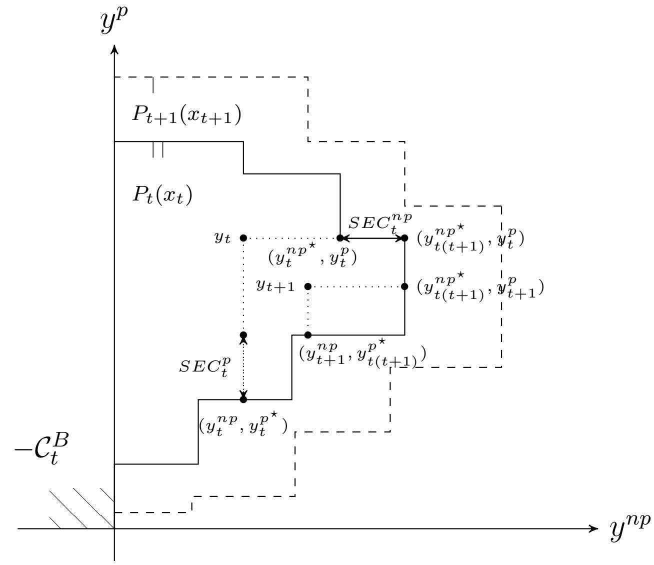

Figure 10: Environmental scale efficiency change of non convex output set $(t, t+1)$

Figure 10 shows that SEC_t^{np} is the distance between $(y^{np\star}_{t(t+1)}, y^p_t)$ and $(y_t^{np\star}, y_t^p)$ in no polluting direction. The same reasoning is applied for the polluting scale efficiency change of period (t). SEC_t^{np} (SEC_t^p) evaluates changes of no polluting (polluting) outputs along the (t)'s B-disposal boundary when polluting (no polluting) outputs are fixed with respect to y_t and y_{t+1}. $SEC_t^{np} > 0$ ($SEC_t^p > 0$) depicts a relation between the reduction (increase) of polluting (no polluting) outputs in $(t+1)$ and the improvement (decrease) of no polluting (polluting) outputs along the B-disposal boundary of period (t). Therefore, $ESEC_t > 0$ shows global environmental scale efficiency improvement for time period (t). Table 3 summarizes the conditions of environmental scale efficiency variation.

	$SEC_t^p > 0$	$SEC_t^p < 0$
$SEC_t^{np} > 0$	$ESEC_t > 0$	**i.** $\lvert SEC_t^{np}\rvert > \lvert SEC_t^p\rvert$ then $ESEC_t > 0$, **ii.** $\lvert SEC_t^{np}\rvert < \lvert SEC_t^p\rvert$ then $ESEC_t < 0$
$SEC_t^{np} < 0$	**i.** $\lvert SEC_t^{np}\rvert < \lvert SEC_t^p\rvert$ then $ESEC_t > 0$ **ii.** $\lvert SEC_t^{np}\rvert < \lvert SEC_t^p\rvert$ then $ESEC_t < 0$	$ESEC_t < 0$

Table 3: Characterization of environmental scale efficiency change

4.2 Environmental Hicks-Moorsteen productivity index

Multiplicative decomposition (Diewert and Fox, 2017) of the environmental Hicks-Moorsteen productivity index is defined as follows.

Definition 4.3 *Let $P_t(x_t)$ be a B-disposal output set that satisfies properties $P1 - P4$. For any consecutive time periods $(t, t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, with $y_{t,t+1} = (y^{np}_{t,t+1}, y^p_{t,t+1}) \in \mathbb{R}^m_+$, the global environmental Hicks-Moorsteens index can be* *decomposed as follows:*

$$EHM_{t,t+1} = \mathcal{ETC}_{t,t+1} \times \mathcal{EEV}_{t,t+1} \times \mathcal{ESEC}_{t,t+1}. \tag{4.17}$$

Where,

i. $\mathcal{ETC}_{t,t+1}$ *shows environmental technical change among the periods* (t) *and* $(t+1)$.

ii. $\mathcal{EEV}_{t,t+1}$ *depicts environmental efficiency variation among the periods* (t) *and* $(t+1)$.

iii. $\mathcal{ESEC}_{t,t+1}$ *allows to identify environmental scale efficiency change among the periods* (t) *and* $(t+1)$.

Environmental technical change among the periods (t) and $(t+1)$ is defined below :

$$\mathcal{ETC}_{t,t+1} = \mathcal{TC}^{np}_{t,t+1} \times \mathcal{TC}^{p}_{t,t+1}. \tag{4.18}$$

Such that,

$$\mathcal{TC}^{np}_{t,t+1} = \Big[\frac{\psi^{np}_t(x_t, y^{np}_t, y^p_t)}{\psi^{np}_{t+1}(x_{t+1}, y^{np}_t, y^p_{t+1})} \times \frac{\psi^{np}_t(x_t, y^{np}_{t+1}, y^p_t)}{\psi^{np}_{t+1}(x_{t+1}, y^{np}_{t+1}, y^p_{t+1})}\Big]^{\frac{1}{2}} \tag{4.19}$$

and

$$\mathcal{TC}^{p}_{t,t+1} = \Big[\frac{\psi^{p}_t(x_t, y^{np}_t, y^p_t)}{\psi^{p}_{t+1}(x_{t+1}, y^{np}_{t+1}, y^p_t)} \times \frac{\psi^{p}_t(x_t, y^{np}_t, y^p_{t+1})}{\psi^{p}_{t+1}(x_{t+1}, y^{np}_{t+1}, y^p_{t+1})}\Big]^{\frac{1}{2}} \tag{4.20}$$

are respectively no polluting and polluting technical change among time periods (t) and $(t+1)$. If $\mathcal{TC}^{np}_{t,t+1} > 1$ ($\mathcal{TC}^{p}_{t,t+1} > 1$) then, production frontier shifts to the right (downwards). Hence, $\mathcal{ETC}_{t,t+1}$ allows to define technical change in both polluting and no polluting directions. If $\mathcal{ETC}_{t,t+1} > 1$ then, global environmental technical advance arises. The combination of the measures of $\mathcal{TC}^{np}_{t,t+1}$ and $\mathcal{TC}^{p}_{t,t+1}$ gives informations about the change of the B-disposal boundary. Table 4 summarizes the conditions of environmental technical change characterization.

	$\mathcal{TC}^{p}_{t,t+1} > 1$	$\mathcal{TC}^{p}_{t,t+1} < 1$
$\mathcal{TC}^{np}_{t,t+1} > 1$	$\mathcal{ETC}_{t,t+1} > 1$, shift to the right and to the downwards	**i.** $\mathcal{TC}^{np}_{t,t+1} > \Big[\mathcal{TC}^{p}_{t,t+1}\Big]^{-1}$ then $\mathcal{ETC}_{t,t+1} > 1$, shift to the right and to the upwards **ii.** $\mathcal{TC}^{np}_{t,t+1} < \Big[\mathcal{TC}^{p}_{t,t+1}\Big]^{-1}$ then $\mathcal{ETC}_{t,t+1} < 1$, shift to the right and to the upwards
$\mathcal{TC}^{np}_{t,t+1} < 1$	**i.** $\Big[\mathcal{TC}^{np}_{t,t+1}\Big]^{-1} < \mathcal{TC}^{p}_{t,t+1}$ then $\mathcal{ETC}_{t,t+1} > 1$, shift to the left and to the downwards **ii.** $\Big[\mathcal{TC}^{np}_{t,t+1}\Big]^{-1} > \mathcal{TC}^{p}_{t,t+1}$ then $\mathcal{ETC}_{t,t+1} < 1$, shift to the left and to the downwards	$\mathcal{ETC}_{t,t+1} < 1$, shift to the left and to the upwards

Table 4: $\mathcal{ETC}_{t,t+1}$ characterization

Environmental efficiency variation among periods (t) and $(t+1)$ is defined as follows:

$$\mathcal{EEV}_{t,t+1} = \mathcal{EV}^{np}_{t,t+1} \times \mathcal{EV}^{p}_{t,t+1} \tag{4.21}$$

Where,

$$\mathcal{EV}^{np}_{t,t+1} = \frac{\psi^{np}_{t+1}(x_{t+1}, y^{np}_{t+1}, y^{p}_{t+1})}{\psi^{np}_{t}(x_{t}, y^{np}_{t}, y^{p}_{t})} \tag{4.22}$$

and

$$\mathcal{EV}^{p}_{t,t+1} = \frac{\psi^{p}_{t+1}(x_{t+1}, y^{np}_{t+1}, y^{p}_{t+1})}{\psi^{p}_{t}(x_{t}, y^{np}_{t}, y^{p}_{t})} \tag{4.23}$$

show no polluting and polluting efficiency change over time periods (t) and $(t+1)$. If $\mathcal{EV}^{np}_{t,t+1} > 1$ ($\mathcal{EV}^{p}_{t,t+1} > 1$) then, efficiency increases in no polluting (polluting) direction among periods (t) and $(t+1)$. Hence, $\mathcal{EEV}_{t,t+1} > 1$ allows to define global environmental efficiency improvement over time $(t, t+1)$. Table 5 presents the conditions of environmental efficiency variation.

	$\mathcal{EV}^{p}_{t,t+1} > 1$	$\mathcal{EV}^{p}_{t,t+1} < 1$
$\mathcal{EV}^{np}_{t,t+1} > 1$	$\mathcal{EEV}_{t,t+1} > 1$	**i.** $\mathcal{EV}^{np}_{t,t+1} > \left[\mathcal{EV}^{p}_{t,t+1}\right]^{-1}$ then $\mathcal{EEV}_{t,t+1} > 1$, **ii.** $\mathcal{EV}^{np}_{t,t+1} < \left[\mathcal{EV}^{p}_{t,t+1}\right]^{-1}$ then $\mathcal{EEV}_{t,t+1} < 1$
$\mathcal{EV}^{np}_{t,t+1} < 1$	**i.** $\left[\mathcal{EV}^{np}_{t,t+1}\right]^{-1} < \mathcal{EV}^{p}_{t,t+1}$ then $\mathcal{EEV}_{t,t+1} > 1$ **ii.** $\left[\mathcal{EV}^{np}_{t,t+1}\right]^{-1} > \mathcal{EV}^{p}_{t,t+1}$ then $\mathcal{EEV}_{t,t+1} < 1$	$\mathcal{EEV}_{t,t+1} < 1$

Table 5: $\mathcal{EEV}_{t,t+1}$ characterization

Environmental scale efficiency change among periods (t) and $(t+1)$ is defined as,

$$\mathcal{ESEC}_{t,t+1} = EHM_{t,t+1} \times \left(\mathcal{ETC}_{t,t+1} \times \mathcal{EEV}_{t,t+1}\right)^{-1}. \tag{4.24}$$

Result (4.24) shows that if there are no technical change and no efficiency variation then, environmental productivity growth comes from environmental scale efficiency change. Indeed, if $\mathcal{ETC}_{t,t+1} = \mathcal{EEV}_{t,t+1} = 1$ then, $\mathcal{ESEC}_{t,t+1} = EHM_{t,t+1}$.

Multiplicative decomposition (Diewert and Fox, 2017) of environmental scale efficiency change is defined below :

Proposition 4.4 *Let $P_t(x_t)$ be an output set that satisfies properties $P1 - P4$ and $P6$. For any consecutive time periods $(t, t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}^{n+m}_{+}$, such that $y_t = (y^{np}_t, y^{p}_t) \in P_t(x_t)$ and $y_{t+1} = (y^{np}_{t+1}, y^{p}_{t+1}) \in P_{t+1}(x_{t+1})$, multiplicative decomposition of scale efficiency change is defined as follows:*

$$\mathcal{ESEC}_{t,t+1} = \left(\mathcal{ESEC}_{t} \times \mathcal{ESEC}_{t+1}\right)^{\frac{1}{2}}. \tag{4.25}$$

$\mathcal{ESEC}_{t}$ and $\mathcal{ESEC}_{t+1}$ respectively show the change in no polluting and polluting outputs along the B-disposal frontier with respect to period (t) and period $(t+1)$, .

Undesirable and desirable outputs change along the (t)'s B-disposal frontier is defined as follows:

$$\mathcal{ESEC}_{t} = \mathcal{SEC}^{np}_{t} \times \mathcal{SEC}^{p}_{t}. \tag{4.26}$$

Where,

$$\mathcal{SEC}_t^{np} = \frac{\psi_t^{np}(x_t, y_t^{np^\star}, y_t^p)}{\psi_t^{np}(x_t, y_{t(t+1)}^{np^\star}, y_t^p)} \tag{4.27}$$

and

$$\mathcal{SEC}_t^{p} = \frac{\psi_t^{p}(x_t, y_t^{np}, y_{t(t+1)}^{p^\star})}{\psi_t^{p}(x_t, y_t^{np}, y_t^{p^\star})} \tag{4.28}$$

No polluting and polluting outputs optimal projection along the (t)'s B-disposal frontier are respectively defined as follows:

$$y_t^{np^\star} = \frac{y_t^{np}}{\psi_t^{np}(x_t, y_t^{np}, y_t^p)} \tag{4.29}$$

$$y_t^{p^\star} = \frac{y_t^{p}}{\psi_t^{p}(x_t, y_t^{np}, y_t^p)} \tag{4.30}$$

and

$$y_{t(t+1)}^{np^\star} = \frac{y_{t+1}^{np}}{\psi_t^{np}(x_{t+1}, y_{t+1}^{np}, y_{t+1}^p)} \tag{4.31}$$

$$y_{t(t+1)}^{p^\star} = \frac{y_{t+1}^{p}}{\psi_t^{p}(x_{t+1}, y_{t+1}^{np}, y_{t+1}^p)}. \tag{4.32}$$

If $\mathcal{SEC}_t^{np} > 1$ ($\mathcal{SEC}_t^{p} > 1$) then, the scale efficiency increases in no polluting (polluting) direction for time (t). Therefore, $\mathcal{ESEC}_t > 1$ allows to define global environmental scale efficiency improvement for time (t). Table 6 summarizes the conditions of environmental scale efficiency variation.

	$\mathcal{SEC}_t^{p} > 1$	$\mathcal{SEC}_t^{p} < 1$
$\mathcal{SEC}_t^{np} > 1$	$\mathcal{ESEC}_t > 1$	**i.** $\mathcal{SEC}_t^{np} > \left[\mathcal{SEC}_t^{p}\right]^{-1}$ then $\mathcal{ESEC}_t > 1$, **ii.** $\mathcal{SEC}_t^{np} < \left[\mathcal{SEC}_t^{p}\right]^{-1}$ then $\mathcal{ESEC}_t < 1$
$\mathcal{SEC}_t^{np} < 1$	**i.** $\left[\mathcal{SEC}_t^{np}\right]^{-1} < \mathcal{SEC}_t^{p}$ then $\mathcal{ESEC}_t > 1$ **ii.** $\left[\mathcal{SEC}_t^{np}\right]^{-1} > \mathcal{SEC}_t^{p}$ then $\mathcal{ESEC}_t < 1$	$\mathcal{ESEC}_t < 1$

Table 6: $\mathcal{ESEC}_t$ characterization

5 Environmental productivity measures on non-parametric technologies

In this section, we focus on convex and non-convex non-parametric technologies. The new environmental efficiency measures are defined through the so-called Data Envelopment Analysis (DEA) model (Banker, Charnes and Cooper, 1984) and the Free Disposal Hull (FDH) non-convex production model (Tulkens, 1993).

5.1 Non-parametric convex and non-convex B-disposal technologies

Let us consider the following notation : $(x_t, y_t) = (x, y)$ et $(x_{t+1}, y_{t+1}) = (\hat{x}, \hat{y})$. In addition, assume that $\mathcal{A} = \{(x_z, y_z) : z \in \mathcal{Z}\}$ is a set of Decision Making Units (DMUs), such that $\mathcal{Z}$ is an index set of natural number. For any $(x_0, y_0) \in \mathcal{A}$, non-parametric convex B-disposal output set (Abad and Briec, 2019) of period (t) is defined as follows (Figures 11-13): $P_t^{\{\emptyset,B\},DEA}(x_0) = P_t^{\emptyset,DEA}(x_0) \cap P_t^{B,DEA}(x_0) = \Big(\big(P_t^{DEA}(x_0) - \mathbb{R}_+^m\big) \cap \big(P_t^{DEA}(x_0) - \mathcal{C}_t^B\big) \Big) \cap \mathbb{R}_+^m$. Therefore, we have:

$$\begin{aligned} P_t^{\{\emptyset,B\},DEA}(x_0) = \Big\{ y : x_{0,i} &\geq \sum_{z \in \mathcal{Z}} \theta_z x_{z,i}, \;\; i = 1, ..., n \\ x_{0,i} &\geq \sum_{z \in \mathcal{Z}} \mu_z x_{z,i}, \;\; i = 1, ..., n \\ y_j &\geq \sum_{z \in \mathcal{Z}} \theta_z y_{z,j}, \;\; j \in B \\ y_j &\leq \sum_{z \in \mathcal{Z}} \theta_z y_{z,j}, \;\; j \notin B \\ y_j &\leq \sum_{z \in \mathcal{Z}} \mu_z y_{z,j}, \;\; j = 1, ..., m \\ \sum_{z \in \mathcal{Z}} \theta_z &= \sum_{z \in \mathcal{Z}} \mu_z = 1, \;\; \theta, \mu \geq 0 \Big\}. \end{aligned} \tag{5.1}$$

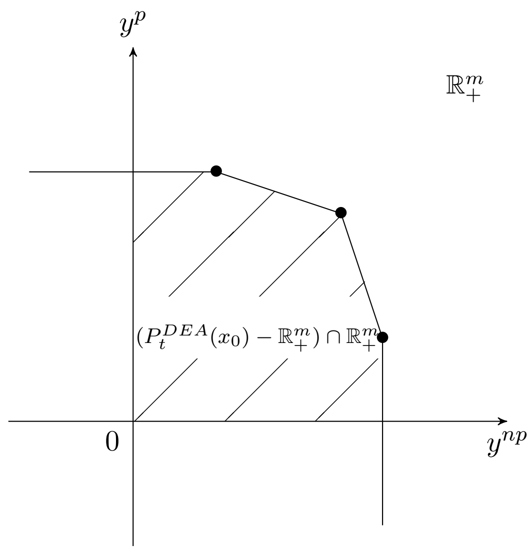

Figure 11: Subset $(P_t^{DEA}(x_0) - \mathbb{R}_+^m) \cap \mathbb{R}_+^m$

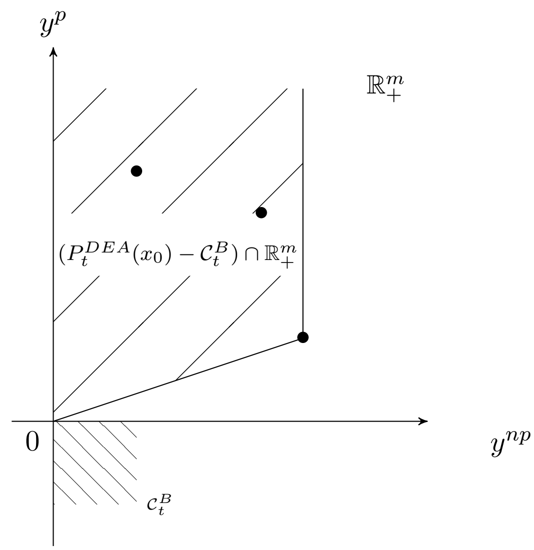

Figure 12: Subset $(P_t^{DEA}(x_0) - \mathcal{C}_t^B) \cap \mathbb{R}_+^m$

For any $z \in \mathcal{Z}$, let us introduce the following individual production possibility set:

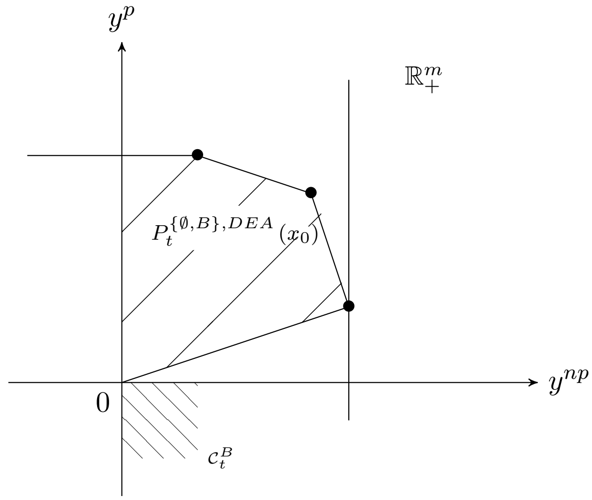

Figure 13: Non-parametric convex B-disposal output set ($P1 - P6$)

$$S^{\emptyset}(x_z, y_z) = \Big\{(x,y) \in \mathbb{R}^{n+m}_+ : x_i \geq x_{z,i},\ i = 1, ..., n \\ y_j \leq y_{z,j},\ j = 1, ..., m\Big\} \tag{5.2}$$

and

$$S^{B}(x_z, y_z) = \Big\{(x,y) \in \mathbb{R}^{n+m}_+ : x_i \geq x_{z,i},\ i = 1, ..., n \\ y_j \leq y_{z,j},\ j \notin B \\ y_j \geq y_{z,j},\ j \in B\Big\}. \tag{5.3}$$

FDH non-convex B-disposal output set (Abad and Briec, 2019) of period (t) is defined as follows (Figures 14-16):

$$P_{nc}^{\{\emptyset,B\},DEA}(x) = \Big\{y : (x,y) \in \big(\cup_{z\in\mathcal{Z}} S^{\emptyset}(x_z, y_z)\big) \cap \big(\cup_{z\in\mathcal{Z}} S^{B}(x_z, y_z)\big)\Big\}. \tag{5.4}$$

5.2 Non-parametric convex and non convex environmental productivity measures

For any $(x_z, y_z) \in \mathcal{A}$, distance functions within the environmental productivity measures are computed from convex or non convex non-parametric B-disposal output set.

5.2.1 Environmental generalized multiplicative efficiency measure

From the specification of convex non-parametric B-disposal output set, proposition below introduces an environmental generalized multiplicative distance function.

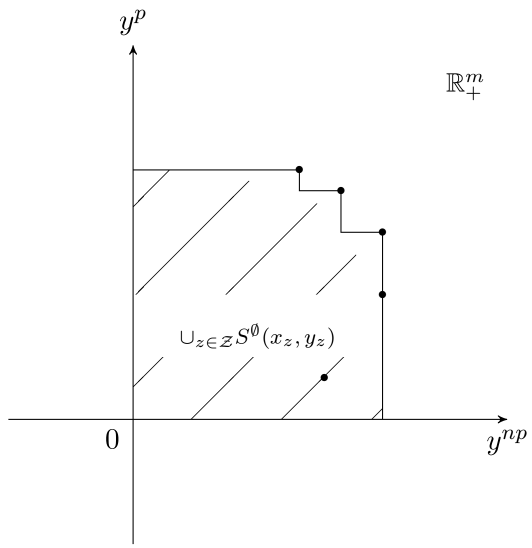

Figure 14: Non convex union of 5.2

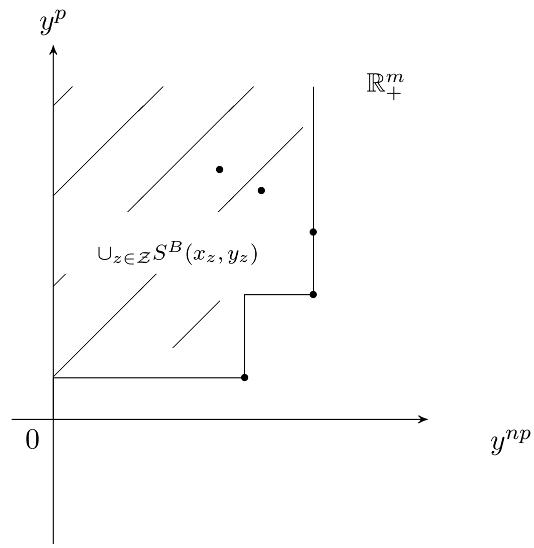

Figure 15: Non convex union of 5.3

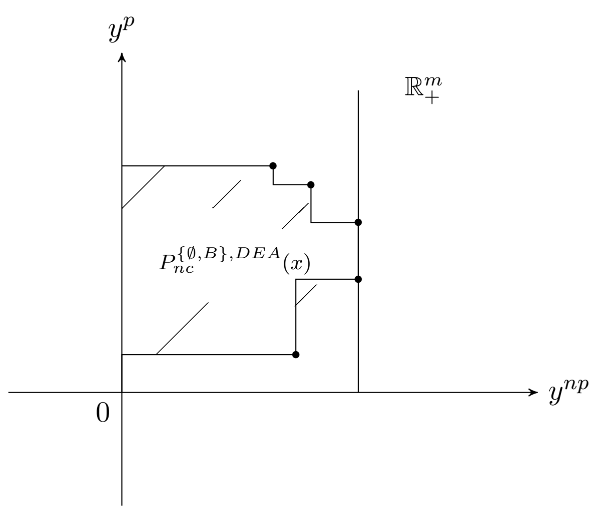

Figure 16: FDH non-convex B-disposal output set ($P1 - P4$ and $P6$)

Proposition 5.1 *Let $P_t(x_t)$ be a B-disposal output set that satisfies properties $P1-P5$. For any consecutive time periods $(t,t+1)$ and for any $(x_{t,t+1},y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, with $y_{t,t+1} = (y_{t,t+1}^{np}, y_{t,t+1}^{p}) \in \mathbb{R}_+^m$, the non-parametric environmental generalized multiplicative efficiency measure is the solution of the following mathematical program:*

$$
\begin{aligned}
\psi_t^{DEA}(x_0, y_0^{np}, y_0^p) = \inf \lambda & \\
s.t.\ x_{0,i} &\geq \sum_{z\in\mathcal{Z}} \theta_z x_{z,i},\ \ i=1,...,n \\
x_{0,i} &\geq \sum_{z\in\mathcal{Z}} \mu_z x_{z,i},\ \ i=1,...,n \\
\lambda^{\beta^p} y_{0,j}^p &\geq \sum_{z\in\mathcal{Z}} \theta_z y_{z,j},\ \ j\in B \\
\lambda^{\beta^{np}} y_{0,j}^{np} &\leq \sum_{z\in\mathcal{Z}} \theta_z y_{z,j},\ \ j\notin B \\
\lambda^{\beta^p} y_{0,j}^p &\leq \sum_{z\in\mathcal{Z}} \mu_z y_{z,j},\ \ j\in B \\
\lambda^{\beta^{np}} y_{0,j}^{np} &\leq \sum_{z\in\mathcal{Z}} \mu_z y_{z,j},\ \ j\notin B \\
\sum_{z\in\mathcal{Z}} \theta_z &= \sum_{z\in\mathcal{Z}} \mu_z = 1,\ \ \theta,\mu\geq 0
\end{aligned}
\tag{5.5}
$$

with $\beta^p = \{0,1\}$ and $\beta^{np} = \{-1,0\}$ and $(x_t,y_t)=(x,y)$.

The following proposition allows to compute environmental generalized multiplicative distance function on FDH non-convex B-disposal output set.

Proposition 5.2 *Let $P_t(x_t)$ be a B-disposal output set that satisfies properties $P1-P4$. For any consecutive time periods $(t,t+1)$ and for any $(x_{t,t+1},y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, with $y_{t,t+1} = (y_{t,t+1}^{np}, y_{t,t+1}^{p}) \in \mathbb{R}_+^m$, the non-parametric environmental generalized multiplicative efficiency measure of period (t) is defined as follows:*

$$
\psi_{tnc}^{DEA}(x,y) = \begin{cases}
\max\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\dfrac{y_j}{y_{z,j}}\right) & if\ \beta^{np}=-1\ and\ \beta^p = 0 \\
\left\{\begin{array}{ll} \max\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\dfrac{y_j}{y_{z,j}}\right) & if\ \min\limits_{\substack{z\in\mathcal{Z}\\ j\in B}} \left(\dfrac{y_{z,j}}{y_j}\right) \geq \max\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\dfrac{y_j}{y_{z,j}}\right) \\ 1 & else \end{array}\right\} & if\ \beta^{np}=-1\ and\ \beta^p = 1 \\
\max\limits_{\substack{z\in\mathcal{Z}\\ j\in B}} \left(\dfrac{y_{z,j}}{y_j}\right) & if\ \beta^{np}=0\ and\ \beta^p = 1
\end{cases}
\tag{5.6}
$$

where $(x_t,y_t)=(x,y)$.

Proof: Let $\beta^{np} = -1$ and $\beta^p = 1$,

$$\begin{aligned}
\psi_{tnc}^{DEA}(x,y) &= \min_{z\in\mathcal{Z}} \Big\{ \quad \min\Big\{\lambda : x \geq x_z, \quad \frac{y^{np}}{\lambda} \leq y_z^{np}, \lambda y^p \leq y_z^p\Big\}; \\
&\qquad\qquad \min\Big\{\lambda : x \geq x_z, \quad \frac{y^{np}}{\lambda} \leq y_z^{np}, \lambda y^p \geq y_z^p\Big\}\Big\} \\
&= \min_{z\in\mathcal{Z}} \left\{ \quad \min\left\{\lambda : x \geq x_z, \quad \lambda \geq \max_{j\notin B}\left(\frac{y_j}{y_{z,j}}\right), \lambda \leq \min_{j\in B}\left(\frac{y_{z,j}}{y_j}\right)\right\}; \right. \\
&\qquad\qquad \left. \min\left\{\lambda : x \geq x_z, \quad \lambda \geq \max_{j\notin B}\left(\frac{y_j}{y_{z,j}}\right), \lambda \geq \max_{j\in B}\left(\frac{y_{z,j}}{y_j}\right)\right\}\right\}
\end{aligned}$$

such that $(x_t, y_t) = (x, y)$. Hence, if $\min_{j\in B}\left(\frac{y_{z,j}}{y_j}\right) \geq \max_{j\notin B}\left(\frac{y_j}{y_{z,j}}\right)$ then,

$$\psi_{tnc}^{DEA}(x,y) = \max_{j\notin B}\left(\frac{y_j}{y_{z,j}}\right).$$

Naturally,

$$\min_{j\notin B}\left(\frac{y_j}{y_{z,j}}\right) \leq \max_{j\notin B}\left(\frac{y_j}{y_{z,j}}\right) \Leftrightarrow \max_{j\notin B}\left(\frac{y_{z,j}}{y_j}\right) \leq \min_{j\notin B}\left(\frac{y_{z,j}}{y_j}\right).$$

Therefore, if $\min_{j\in B}\left(\frac{y_{z,j}}{y_j}\right) \geq \max_{j\notin B}\left(\frac{y_j}{y_{z,j}}\right)$ then, $\min_{j=1,\dots,m}\left(\frac{y_{z,j}}{y_j}\right) \geq \max_{j\notin B}\left(\frac{y_j}{y_{z,j}}\right) \Leftrightarrow \max_{j=1,\dots,m}\left(\frac{y_{z,j}}{y_j}\right) \geq \max_{j\notin B}\left(\frac{y_j}{y_{z,j}}\right)$. Consequently, $\min_{j\in B}\left(\frac{y_{z,j}}{y_j}\right) \geq \max_{j\notin B}\left(\frac{y_j}{y_{z,j}}\right)$ is always true for at least the DMU evaluated relatively to itself. If $\min_{j\in B}\left(\frac{y_{z,j}}{y_j}\right) < \max_{j\notin B}\left(\frac{y_j}{y_{z,j}}\right)$ occurs for the remaining DMUs such that there does not exist $\lambda < 1$ with $\lambda \in \left[\max_{j\notin B}\left(\frac{y_j}{y_{z,j}}\right); \max_{j\in B}\left(\frac{y_{z,j}}{y_j}\right)\right]$ then,

$$\begin{aligned}
\min_{z\in\mathcal{Z}} &\left\{ \min\left\{\lambda : x \geq x_z, \lambda \geq \max_{j\notin B}\left(\frac{y_j}{y_{z,j}}\right), \lambda \leq \min_{j\in B}\left(\frac{y_{z,j}}{y_j}\right)\right\}; \right. \\
&\left. \min\left\{\lambda : x \geq x_z, \lambda \geq \max_{j\notin B}\left(\frac{y_j}{y_{z,j}}\right), \lambda \geq \max_{j\in B}\left(\frac{y_{z,j}}{y_j}\right)\right\}\right\} = 1
\end{aligned}$$

Indeed, a DMU evaluated relatively to itself is always efficient; ie. its efficiency score is equal to 1. Therefore,

$$\psi_{tnc}^{DEA}(x,y) = 1.$$

The proof for $\beta^{np} = -1$ and $\beta^p = 0$ ($\beta^{np} = 0$ and $\beta^p = 1$) can be directly deduced from the proof of $\beta^{np} = -1$ and $\beta^p = 1$. $\square$

The following corollary presents equivalence conditions for the non-parametric environmental generalized multiplicative distance function, non-parametric desirable and undesirable output-oriented Shephard distance functions (Färe et al., 2004), and the non-parametric hyperbolic output efficiency measure (Färe et al., 1989).

Corollary 5.3 *Let $P_t^{\{\emptyset,B\},DEA}(x_t)$ be a convex non-parametric B-disposal output set and assume that $\theta = \mu \in P_t^{\{\emptyset,B\},DEA}(x_t)$. For any $(x_t, y_t) \in \mathbb{R}_+^{n+m}$, with $y_t = (y_t^{np}, y_t^p) \in P_t^{\{\emptyset,B\},DEA}(x_t)$, we have:*

i. $\psi_t^{DEA}(x_t, y_t) \equiv D_t^{np,DEA}(x_t, y_t)$, *with $\beta^p = 0$ and $\beta^{np} = -1$.*

ii. $\psi_t^{DEA}(x_t, y_t) \equiv D_t^{p,DEA}(x_t, y_t)$, *with $\beta^p = 1$ and $\beta^{np} = 0$.*

iii. $\psi_t^{DEA}(x_t, y_t) \equiv H_t^{o,DEA}(x_t, y_t)$, *with $\beta^p = 1$ and $\beta^{np} = -1$.*

If $\theta = \mu \in P_t^{\{\emptyset,B\},DEA}(x_t)$ then, convex non-parametric B-disposal output set provides a characterization of incorrect modelling of VRS assumption in traditional Shephard's WD technologies (Abad and Briec, 2019; Abad and Ravelojaona, 2017). Therefore, the above results are immediate.

5.2.2 Environmental generalized additive efficiency measure

For a convex non-parametric B-disposal output set, the following proposition introduces non-parametric environmental generalized additive distance function.

Proposition 5.4 *Let $P_t(x_t)$ be a B-disposal output set that satisfies properties $P1 - P5$. For any consecutive time periods $(t, t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, with $y_{t,t+1} = (y_{t,t+1}^{np}, y_{t,t+1}^p) \in \mathbb{R}_+^m$, the non-parametric environmental generalized additive efficiency measure is the solution of the following mathematical program:*

$$
\begin{aligned}
\xi_t^{\{0,\sigma\},DEA}(x_0, y_0) = \max \beta & \\
s.t.\ & x_{0,i} \geq \sum_{z\in\mathcal{Z}} \theta_z x_{z,i}, \quad i = 1, ..., n \\
& x_{0,i} \geq \sum_{z\in\mathcal{Z}} \mu_z x_{z,i}, \quad i = 1, ..., n \\
& (1 + \beta \odot \sigma^p) y_{0,j} \geq \sum_{z\in\mathcal{Z}} \theta_z y_{z,j}, \quad j \in B \\
& (1 + \beta \odot \sigma^{np}) y_{0,j} \leq \sum_{z\in\mathcal{Z}} \theta_z y_{z,j}, \quad j \notin B \\
& (1 + \beta \odot \sigma^p) y_{0,j} \leq \sum_{z\in\mathcal{Z}} \mu_z y_{z,j}, \quad j \in B \\
& (1 + \beta \odot \sigma^{np}) y_{0,j} \leq \sum_{z\in\mathcal{Z}} \mu_z y_{z,j}, \quad j \notin B \\
& \sum_{z\in\mathcal{Z}} \theta_z = \sum_{z\in\mathcal{Z}} \mu_k = 1, \quad \theta, \mu \geq 0
\end{aligned}
\tag{5.7}
$$

with $\sigma = (\sigma^{np}, \sigma^p) \in [0,1]^{m^{np}} \times [-1, 0]^{m^p}$ and $(x_t, y_t) = (x, y)$.

Environmental generalized additive distance function on FDH non-convex B-disposal output set is defined through the proposition below.

Proposition 5.5 *Let $P_t(x_t)$ be a B-disposal output set that satisfies properties $P1-P4$. For any consecutive time periods $(t, t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, with $y_{t,t+1} = (y_{t,t+1}^{np}, y_{t,t+1}^{p}) \in \mathbb{R}_+^m$, the non-parametric environmental generalized additive efficiency measure is defined as follows:*

$$\xi_{t_{nc}}^{\{0,\sigma\},DEA}(x,y) = \begin{cases} \dfrac{1}{y^{np}} \min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(y_{z,j} - y_j\Big) & if\ \sigma = (1,0) \\ \left.\begin{cases} \dfrac{1}{y^{np}} \min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(y_{z,j} - y_j\Big) & if\ \dfrac{1}{y^{p}} \max\limits_{\substack{z\in\mathcal{Z}\\ j\in B}} \Big(y_j - y_{z,j}\Big) \\ & \quad \leq \dfrac{1}{y^{np}} \min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(y_{z,j} - y_j\Big) \\ 0 & else \end{cases}\right\} & if\ \sigma = (1,-1) \\ \dfrac{1}{y^{p}} \min\limits_{\substack{z\in\mathcal{Z}\\ j\in B}} \Big(y_j - y_{z,j}\Big) & if\ \sigma = (0,-1) \end{cases}$$

where $(x_t, y_t) = (x, y)$.

Proof: Let $\sigma = (1, -1)$, we have:

$$\begin{aligned} \xi_{t_{nc}}^{\{0,\sigma\},DEA}(x,y) &= \min_{z\in\mathcal{Z}} \Big\{ \max\Big\{\beta : x \geq x_z,\ \ y^{np} + \beta y^{np} \leq y_z^{np}, y^p - \beta y^p \leq y_z^p\Big\}; \\ &\qquad \max\Big\{\beta : x \geq x_z,\ \ y^{np} + \beta y^{np} \leq y_z^{np}, y^p - \beta y^p \geq y_z^p\Big\}\Big\} \\ &= \min_{z\in\mathcal{Z}} \Bigg\{ \max\Big\{\beta : x \geq x_z,\ \ \beta \leq \frac{1}{y^{np}} \min_{j\notin B} \Big(y_{z,j} - y_j\Big), \\ &\qquad \beta \geq \frac{1}{y^{p}} \max_{j\in B} \Big(y_j - y_{z,j}\Big)\Big\}; \\ &\qquad \max\Big\{\beta : x \geq x_z,\ \ \beta \leq \frac{1}{y^{np}} \min_{j\notin B} \Big(y_{z,j} - y_j\Big), \\ &\qquad \beta \leq \frac{1}{y^{p}} \min_{j\in B} \Big(y_j - y_{z,j}\Big)\Big\}\Bigg\} \end{aligned}$$

where $(x_t, y_t) = (x, y)$. Hence, if $\frac{1}{y^{np}} \min_{j\notin B} \Big(y_{z,j} - y_j\Big) \geq \frac{1}{y^{p}} \max_{j\in B} \Big(y_j - y_{z,j}\Big)$ then,

$$\xi_{t_{nc}}^{\{0,\sigma\},DEA}(x,y) = \frac{1}{y^{np}} \min_{j\notin B} \Big(y_{z,j} - y_j\Big).$$

Naturally,

$$\frac{1}{y^{p}} \min_{j\in B} \Big(y_j - y_{z,j}\Big) \leq \frac{1}{y^{p}} \max_{j\in B} \Big(y_j - y_{z,j}\Big) \Leftrightarrow \frac{1}{y^{p}} \min_{j\in B} \Big(y_{z,j} - y_j\Big) \geq \frac{1}{y^{p}} \max_{j\in B} \Big(y_{z,j} - y_j\Big).$$

Therefore, if $\frac{1}{y^{np}} \min_{j\notin B} \Big(y_{z,j} - y_j\Big) \geq \frac{1}{y^{p}} \max_{j\in B} \Big(y_j - y_{z,j}\Big)$ then, $\frac{1}{y} \min_{j=1,\dots,m} \Big(y_{z,j} - y_j\Big) \geq \frac{1}{y^{p}} \max_{j\in B} \Big(y_j - y_{z,j}\Big) \Leftrightarrow \frac{1}{y^{p}} \min_{j\in B} \Big(y_{z,j} - y_j\Big) \leq \frac{1}{y} \min_{j=1,\dots,m} \Big(y_{z,j} - y_j\Big).$

Consequently, $\frac{1}{y^{np}} \min_{j \notin B} \Big(y_{z,j} - y_j \Big) \geq \frac{1}{y^{p}} \max_{j \in B} \Big(y_j - y_{z,j} \Big)$ is always true for at least the DMU evaluated relatively to itself. If $\frac{1}{y^{np}} \min_{j \notin B} \Big(y_{z,j} - y_j \Big) < \frac{1}{y^{p}} \max_{j \in B} \Big(y_j - y_{z,j} \Big)$ occurs for the remaining DMUs such that there does not exist $\beta > 0$ with $\beta \in \left[\frac{1}{y^{p}} \max_{j \in B} \Big(y_{z,j} - y_j \Big); \frac{1}{y^{np}} \min_{j \notin B} \Big(y_{z,j} - y_j \Big) \right]$ then,

$$\max \Big\{ \beta : x \geq x_z, \beta \leq \frac{1}{y^{np}} \min_{j \notin B} \Big(y_{z,j} - y_j \Big), \beta \leq \frac{1}{y^{p}} \min_{j \in B} \Big(y_j - y_{z,j} \Big) \Big\} = 0.$$

Indeed, a DMU evaluated relatively to itself is always efficient; ie. its efficiency score is equal to 0. Therefore,

$$\begin{aligned} \xi_{tnc}^{\{0,\sigma\},DEA}(x,y) = \min_{z \in \mathcal{Z}} \Bigg\{ \max \Big\{ \beta : \quad & x \geq x_z, \beta \leq \frac{1}{y^{np}} \min_{j \notin B} \Big(y_{z,j} - y_j \Big), \\ & \beta \geq \frac{1}{y^{p}} \max_{j \in B} \Big(y_j - y_{z,j} \Big) \Big\}; 0 \Bigg\} \\ = 0. \end{aligned}$$

The proof for $\sigma = (0, -1)$ and $\sigma = (1, 0)$ can be directly deduced from the proof of $\sigma = (1, -1)$. $\square$

Equivalence conditions among non-parametric environmental generalized additive efficiency measure, non-parametric environmental directional distance function (Chung et al., 1997) and, non-parametric desirable and undesirable sub-vector directional distance functions (Picazo-Tadeo et al., 2014) are introduced below.

Corollary 5.6 *Let $P_t^{\{\emptyset,B\},DEA}(x_t)$ be a convex non-parametric B-disposal output set and assume that $\theta = \mu \in P_t^{\{\emptyset,B\},DEA}(x_t)$. For any $(x_t, y_t) \in \mathbb{R}_+^{n+m}$, with $y_t = (y_t^{np}, y_t^{p}) \in P_t^{\{\emptyset,B\},DEA}(x_t)$, we have:*

i. $\xi_t^{\{0,\sigma\},DEA}(x_t, y_t) \equiv \overrightarrow{D}_t^{np,DEA}(x_t, y_t; y_t^{np}, 0)$, *with* $\sigma = (\sigma^{np}, \sigma^{p}) = (1, 0)$.

ii. $\xi_t^{\{0,\sigma\},DEA}(x_t, y_t) \equiv \overrightarrow{D}_t^{p,DEA}(x_t, y_t; 0, -y_t^{p})$, *with* $\sigma = (\sigma^{np}, \sigma^{p}) = (0, -1)$.

iii. $\xi_t^{\{0,\sigma\},DEA}(x_t, y_t) \equiv \overrightarrow{D}_t^{DEA}(x_t, y_t; 0, y_t^{np}, -y_t^{p})$, *with* $\sigma = (\sigma^{np}, \sigma^{p}) = (1, -1)$.

6 Concluding Comments

This paper gives a more general representation of environmental issues in production economics. Consecutively, environmental generalized efficiency measures and green growth productivity are analysed. New generalization of environmental efficiency measures admits as specific cases usual green efficiency concepts. Equivalence conditions among traditional scheme and a new environmental efficiency analysis framework are proposed.

Generalized environmental productivity indices and indicators inherit the basic structures of Hicks-Moorsteen and Luenberger-Hicks-Moorsteen measures. No polluting sub-vector measures concentrate on desirable components (no polluting outputs). Reversely, polluting sub-vector environmental productivity measures focus on polluting outputs. Consequently, these productivity indices and indicators allow to measure the part of quality changes on productivity growth (or loss). This could be of a particular concern for firms, policy makers or researchers interested in environmental empirical studies.

Knowing core components of green growth productivity variation is a major concern to define global environmental recommendations (public or private). Hence, this paper decomposes (Diewert and Fox, 2017; Ang and Kerstens, 2017) generalized environmental productivity measures into green components of technical change, efficiency variation and scale efficiency change. In addition, our general approach does not require to assume convexity of the production technology. This position have some theoretical and empirical implications. Following a non-parametric framework, a procedure to define general environmental productivity is defined. An extension of this theoretical paper could be the presentation of an empirical application. Such investigation is left for future research.

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